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MATHEMATICS

FOR

ENGINEERING STUDENTS

BY

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CARNEGIE TECHNICAL SCHOOLS

PLANE AND SOLID GEOMETRY

SECOND EDITION, REVISED



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P R E F A C E .

IT has been the author's endeavor in writing this book on Geometry to put the student at ease as far as possible by appealing from the start to his common sense.

Technicalities have been avoided wherever possible and the student encouraged to think about the propositions in the same, simple, common sense way that he would consider any practical question arising in his daily experience.

This process is applied even to the construction of figures, his ingenuity being called upon for suggestions as to auxiliary lines, etc. In this way, the solution of original propositions is gradually approached, until confidence is acquired.

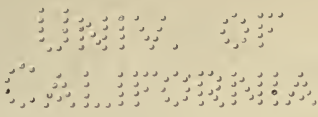
In this work, as in all others, the author has adhered to his conviction that the difficulty in mastering mathematical truths is largely due to the student's awe of the subject, and that any legitimate means of securing confidence and of removing apprehension should be employed.

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PLANE GEOMETRY

DEFINITIONS.

1. GEOMETRY is a study of position, form, and dimension.
2. In geometry we consider points, lines, surfaces, solids, and angles.

A geometrical point has only position; no length nor breadth. Can you make a geometrical point?

Geometrical lines have only length; no width nor thickness. Can you draw a geometrical line? Give a definition of a straight line, and of a curved line.

Solids have length, breadth, and height or thickness. Can you draw a solid?

The boundary that separates a solid from its surroundings is called a surface. A surface has extent, but no thickness.

3. A *surface* is called a *plane* when a straight line joining any two points in it, lies wholly within the surface. Give an illustration of a surface and of a plane.

4. Plane geometry deals only with figures lying in the same plane.

5. An angle is the amount of divergence of two lines drawn through the same point. The point is called the *vertex* of the angle and the two lines, its *sides*.

6. A segment of a line is any portion of it.

When a line is divided into two equal segments, it is said to be *bisected*. When cut into any two segments it is *intersected*.

7. A segment drawn to greater length is said to be *produced*.

8. A *broken* line is made up of a number of straight lines varying in direction.

9. When two straight lines cut each other they form four angles about the point of intersection. The angles situated on the same side of one line and the opposite sides of the other are called *adjacent* angles thus, (see Fig. 1):

ABC and CBD are adjacent $\angle$ s.

ABC and EBD are vertical $\angle$ s.

10. When one straight line makes *equal* adjacent angles with another it is said to be perpendicular to it, and the equal angles are called *right* angles. xyw and wyv (Fig. 2) are right angles.

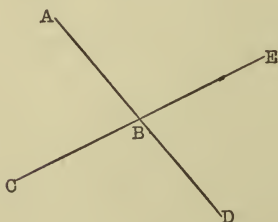


Fig. 1.

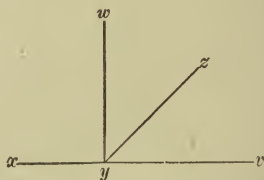


Fig. 2.

11. A *straight* angle is an angle whose sides are in the same straight line on either side of the vertex, thus (Fig. 2): xyv is a straight angle.

Angles are read with the letter at the vertex in the middle,

Thus: angle xyz has y in the middle.

angle zyv has y in the middle, etc.

12. Angles less than a right angle are called *acute*; those

greater, *obtuse*; and collectively, they are known as *oblique* angles. Hence lines meeting at any angle other than a right angle are said to be *oblique* to each other.

13. If two angles together make a right angle, they are said to be *complementary*. If two angles together make a straight angle, they are said to be *supplementary*. wyz and zyv are complementary. xyz and zyv are supplementary (Fig. 2).

AXIOMS.

14. The demonstrations of geometry are based upon certain arbitrary definitions and upon certain truths that are so apparent to average intelligence that they require no proof.

Any argument that arrives at a conclusion must start from some truth or truths accepted by the parties to the argument.

For example, no amount of debate on the phenomena of sound can accomplish any result if one disputant starts from the definition that sound is vibration in the air and the other from the assertion that sound is the effect of vibration on the ear. There must plainly be some common ground of truth from which both must reason. So in geometry certain fundamental assumptions must be agreed upon before reasoning will be effective. Hence the definitions just suggested.

Other definitions would do as well, if they were universally accepted.

Now there happen to be certain truths, in addition to these definitions, that are so evident to everyone that they will be readily adopted. These truths are called axioms and may be stated as follows:

Axiom 1. Things which are equal to the same thing are equal to each other.

Axiom 2. If equals be added to or subtracted from equals, the results will be equal.

Axiom 3. If equals be multiplied or divided by equals, the results are equal.

Axiom 4. If equals be added to or subtracted from unequals, the results are unequal in the same sense.

Axiom 5. If unequals be subtracted from equals, the remainders are unequal in the opposite sense.

Axiom 6. The whole is greater than any of its parts.

Axiom 7. The whole is equal to the sum of all its parts.

Axiom 8. Through two points only one straight line can be drawn.

Axiom 9. The shortest distance between two points is the straight line joining them.

Axiom 10. Magnitudes that can be made to coincide are equal, etc.

GEOMETRICAL PROCESSES.

15. Geometrical processes are of two kinds: demonstration and construction.

Demonstration is the establishment of certain relations between the parts of figures already constructed.

Construction is the actual method of building those figures.

A statement requiring demonstration is called a *theorem*. A statement requiring a construction is called a *problem*. *Proposition* includes both.

A truth so obvious that it does not need proof is called an *axiom* or *postulate*, the latter term applying to purely geometrical truths, and relating more to methods.

A *corollary* is a plain inference from an established theorem.

POSTULATES.

The following truths may be assumed without proof.

It is assumed to be true:

- (a) That a straight line can be drawn from any one point to any other point.
- (b) That a straight line can be produced to any extent.

PERPENDICULAR AND OBLIQUE LINES.

Proposition I. Theorem.

16. *When one straight line crosses another straight line the vertical angles are equal.*

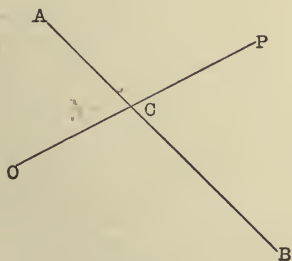


Fig. 3.

Let line OP cross AB at C.

We are to prove $\angle OCB = \angle ACP$.

Analysis: A simple inspection of Fig. 3 will show that both $\angle ACP$ and $\angle OCB$ combine with a common angle $\angle ACO$ to make straight angles, which are always equal.

Proof: $\angle OCA + \angle OCB = 2 \text{ rt. } \angle$.

(Being sup.-adj. $\angle$.)

$\angle OCA + \angle ACP = 2 \text{ rt. } \angle$.

(Being sup.-adj. $\angle$.)

$\therefore \angle OCA + \angle OCB = \angle OCA + \angle ACP$. (Axiom.)

Take away from each of these equals the common $\angle OCA$.

Then, $\angle OCB = \angle ACP$. (Axiom.)

In like manner we may prove,

$\angle OCA = \angle BCP$.

Q.E.D.

Proposition II. Theorem.

17. When the sum of two adjacent angles is equal to two right angles, their exterior sides form one and the same straight line.

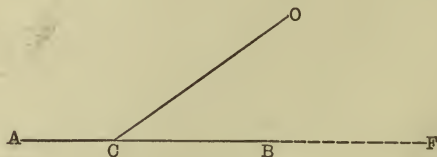


Fig. 4.

Let the adjacent angles $\angle OCA + \angle OCB = 2 \text{ rt. } \angle$.

We are to prove AC and CB in the same straight line.

Analysis: It is evident that if the line AC be extended sup.-adjacent angles will be formed with OC, one of which will be $\angle ACO$. To prove that CB is really this extension, it is only necessary to prove $\angle OCB$ equal to the angle sup.-adjacent to $\angle ACO$.

Proof: Suppose CF to be in the same straight line with AC.

Then $\angle OCA + \angle OCF = 2 \text{ rt. } \angle.$

(Being sup.-adj. $\angle$.)

But $\angle OCA + \angle OCB = 2 \text{ rt. } \angle.$

(By hypothesis.)

$\therefore \angle OCA + \angle OCF = \angle OCA + \angle OCB.$

(Axiom.)

Take away from each of these equals the common $\angle OCA.$

Then $\angle OCF = \angle OCB.$

$\therefore$ CB and CF coincide, and cannot form two distinct lines.

$\therefore$ AC and CB are in the same straight line. *Q.E.D.*

Proposition III. Theorem.

18. *A perpendicular measures the shortest distance from a point to a straight line.*

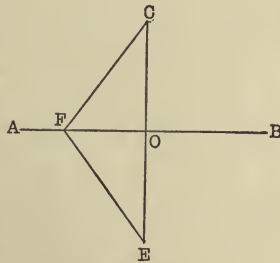


Fig. 5.

Let AB be the given straight line, C the given point, and CO the perpendicular.

We are to prove CO shorter than any other line drawn from C to AB, as CF.

Analysis: We know that a straight line is the shortest distance between two points, hence, if by prolonging each of the lines CO and CF an equal distance, we can reduce the comparison to one between a straight and a broken line, our proposition is easily established.

Produce CO to E, making $OE = OC$. Draw EF.

Proof: On AB as an axis, fold over OCF until it comes into the plane of OEF.

The line OC will take the direction of OE.

(Since $\angle COF = EOF$, each being a rt. $\angle$.)

The point C will fall upon the point E.

(Since $OC = OE$ by cons.)

$\therefore$ Line CF = line FE,

(having their extremities in the same points).

$\therefore CF + FE = 2 CF$,

and $CO + OE = 2 CO$,

But $CO + OE < CF + FE$.

(A straight line is the shortest distance between two points.)

Substitute $2 CO$ for $CO + OE$,

and $2 CF$ for $CF + FE$;

then we have $2 CO < 2 CF$.

$\therefore CO < CF$.

Q.E.D.

Proposition IV. Theorem.

19. *The sum of two lines drawn from a point to the extremities of a straight line is greater than the sum of two other lines similarly drawn, but included by them.*

Let CA and CB be two lines drawn from the point C to the extremities of the straight line AB. Let OA and OB be two lines similarly drawn, but included by CA and CB.

We are to prove $CA + CB > OA + OB$.

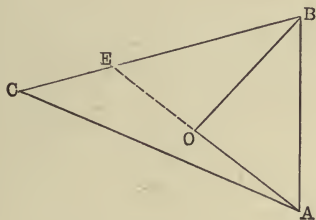


Fig. 6.

Analysis: In comparison of magnitudes, it is always desirable to have some known connecting bond. Hence, if one of the included lines is prolonged until it meets one of the including lines, we have straight lines and broken lines made up of the lines involved in the proposition, which can be readily compared.

Proof: Produce AO to meet the line CB at E.

Then, $AC + CE > AO + OE$,

(A straight line is the shortest distance between two points), and $BE + OE > BO$.

Add these inequalities and we have,

$$AC + CE + BE + OE > OA + OE + OB.$$

Substitute for $CE + BE$ its equal CB , and take away OE from both sides of the inequality, we have,

$$CA + CB > OA + OB.$$

Proposition V.

20. Two oblique lines drawn from the same point on a perpendicular to a straight line, cutting off equal distances

from its foot, are equal; and of two oblique lines, drawn from the same point on a perpendicular, cutting off unequal distances, the more remote is the greater.

Statement: To prove that $MN = MP$, if $NO = OP$, and that $MR > MN$ if $OR > NO$.

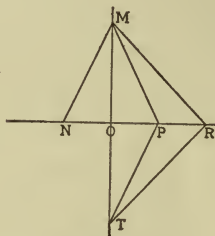


Fig. 7.

Analysis: $\angle MON$ and $\angle MOP$ being right angles enable us to use MO as an axis upon which to revolve $\angle MON$ around upon $\angle MOP$. Also NO being equal to OP , gives a ready method of determining the position of the point N relative to point P .

Proof: (Part 1.) Fold over $\angle MOP$ on MO as an axis. Since $\angle MOP$ and $\angle MON$ are both right angles (because MO is a $\perp$), OP will fall on NO , and since $OP = NO$, the point P will fall on the point N ; hence MP will exactly coincide with MN , since MP and MN have the point M in common, and the point P exactly coinciding with N .
 $\therefore MP = MN$.

(Part 2.) To prove $MR > MN$, if $OR > NO$. Since $MP = MN$, it will be sufficient to prove $MR > MP$.

Analysis. Prop. IV suggests the method of extending the line MO below NR and joining the extremity of this produced line with P and R , thus reproducing the con-

ditions of Prop. IV. It is plainly desirable to make the figure as simple as possible by producing MO a distance OT equal to MO.

Proof: Joining T with P and R, we have

$$\begin{aligned} & \left. \begin{array}{l} PT = PM \\ \text{and } RT = RM \end{array} \right\} \begin{array}{l} \text{(by part 1, since OR is } \perp \text{ to MT)} \\ \text{and MO = OT} \end{array} \\ & \quad RM + RT > PM + PT \quad (\text{Prop. IV.}) \\ \text{or } 2 RM > 2 PM & \quad (\text{since RM = RT and PM = PT.}) \\ & \quad \therefore RM > PM. \end{aligned}$$

That is, $RM > MN$.

21. *Cor. I.* Since in Part I, the two triangles MOP and MON exactly coincided when MOP was folded over upon MON, the $\angle OMP = \angle OMN$; that is, oblique lines drawn from the same point on a perpendicular to a straight line, cutting off equal distances from its foot, make equal angles with the perpendicular.

Proposition VI. Theorem.

22. *If at the middle point of a straight line a perpendicular be erected:*

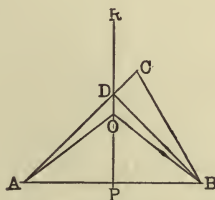


Fig. 8.

(a) *any point in the perpendicular is at equal distances from the extremities of the straight line.*

(b) any point without the perpendicular is at unequal distances from the extremities of the straight line.

Let PR be a perpendicular erected at the middle of the straight line AB, O any point in PR and C any point without PR.

(a) Draw OA and OB. We are to prove $OA = OB$.

Since $PA = PB$, $OA = OB$ (two oblique lines drawn from the same point in a $\perp$, cutting off equal distances from the foot of the $\perp$ are equal.)

(b) Draw CA and CB; C being a point outside PR. We are to prove CA and CB unequal.

One of these lines, as CA, will intersect the $\perp$ at D. From the point of intersection draw DB. $DB = DA$. (Two oblique lines drawn from the same point in a $\perp$, cutting off equal distances from the foot of the $\perp$, are equal.) $CB < CD + DB$ (a straight line is the shortest distance between two points). Substitute for DB its equal DA, then $CB < CD + DA$.

But

$$CD + DA = CA.$$

$$\therefore CB < CA.$$

Q.E.D.

Proposition VII.

23. But one perpendicular can be drawn to a given line through a given point.

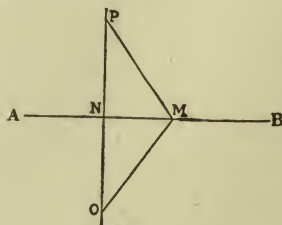


Fig 9.

Statement: PN is the only perpendicular through P to AB.

Analysis: It is known that but one straight line can be drawn, joining two given points; hence by producing PN below AB and joining the extremity with the foot of any other line, presumed to be also a perpendicular, it will be possible to show that PN produced is the only straight line, between the point P and this other extremity.

Proof: Draw any other line PM, to show that it is *not* a perpendicular. Produce PN to O, making $NO = NP$, and join O with M. Then since only one straight line can be drawn between P and O, OMP cannot be a straight line, and hence $\angle PMO$ cannot be a straight angle.

But by Cor. I, Prop. V, $\angle PMN = \angle OMN$ (for AB is $\perp$ to OP).

$$\therefore \angle PMN = \frac{1}{2} \angle PMO.$$

$\therefore$ If $\angle PMO$ is not a straight angle, $\angle PMN$ is not a right angle. Hence PM is not $\perp$ to AB.

Since PM is any line other than PN, PN is the only $\perp$.

Proposition VIII. Theorem.

24. *Two straight lines in the same plane perpendicular to the same straight line are parallel.*

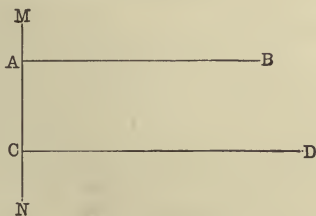


Fig. 10.

Statement: Let AB and CD be both $\perp$ to MN, then are they parallel.

Analysis: Parallel lines can never meet no matter how far produced. If AB and CD do not meet, then they must be parallel.

Proof: If AB and CD should meet when sufficiently prolonged, they would then both extend from that common point of meeting, say E, to the line MN, thus (Fig. 11):

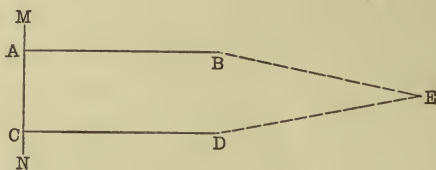


Fig. 11.

and since they are both $\perp$ to MN by hypothesis, we would have two $\perp$'s drawn from the same point to the same straight line, which we know to be impossible, $\therefore$ they cannot meet, and hence are parallel.

Proposition IX.

25. *If a straight line is perpendicular to one of two parallel lines, it is perpendicular to the other also.*

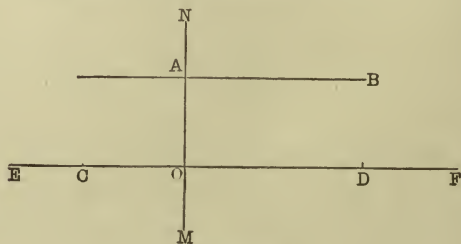


Fig. 12.

Statement: Let MN be $\perp$ to AB, which is parallel to CD, to prove MN $\perp$ to CD

Analysis: A line drawn through $O \perp$ to MN will be $\parallel$ to AB by last proposition. If this line can be identified with CD , our conclusion follows.

Proof: Through O where MN meets CD , draw a line $EF \perp$ to MN , and entirely independent of CD ; then by last proposition AB and EF are $\parallel$. But CD is already drawn through $O \parallel$ to AB by our hypothesis, and since through one point it is impossible to draw more than one line $\parallel$ to another given line, EF and CD must be the same line. But EF was drawn $\perp$ to MN , $\therefore CD$ is $\perp$ to MN .

26. Definition: A line cutting two or more straight lines is called a *transversal*.

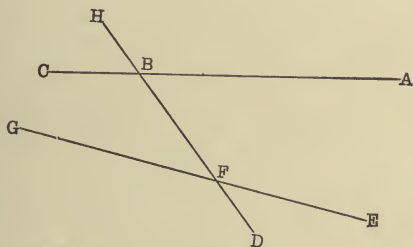


Fig. 13.

HD is a transversal. $\angle ABF$ and $\angle BFG$; also $\angle CBF$ and BFE are called *alternate-interior* angles.

HBC and BFG ; also HBA and BFE are called *exterior-interior* angles. HBC and EFD ; also ABH and DFG are called *alternate-exterior* angles.

Proposition X.

27. If a transversal cuts two parallel lines, the alternate-interior angles thus formed are equal; the interior-exterior

angles are equal and the interior angles on the same side of the transversal are supplementary.

Statement: If a transversal MN cuts the parallels AD and EH, the angle $\text{DBG} = \text{angle BGE}$; the angle $\text{DBM} = \text{angle HGB}$ (or $\text{ABM} = \text{BGE}$) and the angle $\text{DBG} + \text{angle BGH} = \text{two right angles}$, etc.

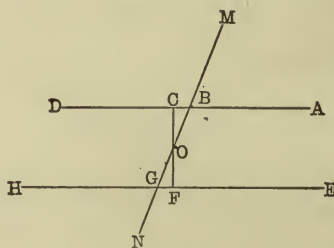


Fig. 14.

Analysis: To prove two angles equal it is necessary first to prove that they will exactly coincide when applied to one another; and to apply them effectively it is best to inclose them in a triangle.

Proof: Draw the auxiliary line CF through the middle point O of BG and $\perp$ to AD, hence $\perp$ to EH. (Why?) What reason is there for drawing CF through the middle of BG?

In order to compare $\angle \text{CBO}$ (DBG) with $\angle \text{OGF}$ (BGE), to which we are to prove it equal, swing the triangle OFG about its vertex O as a pivot, until OG falls directly upon OB. Then, since O was the mid-point of BG, OG must equal OB and the point G will fall exactly upon B, because O is fixed.

Since $\angle FOG = \angle BOC$ (why?), OF will fall upon OC (the other sides of these angles, OG and OB already coinciding).

Since FG and BC, being parts of EH and AD respectively are both $\perp$ to CF (which is simply folded upon itself); since G is now at B and but one perpendicular can be drawn from a given point to a given straight line, FG must fall upon BC, and since two straight lines can meet at but one point, $\therefore$ F must coincide with C, $\therefore$ the two triangles BOC and OFG coincide throughout and $\angle CBO = \angle OGF$, or $\angle DBG = \angle BGE$.

Since $\angle GBD + \angle GBA = 2 \text{ rt. } \angle$,
and $\angle BGE + \angle BGH = 2 \text{ rt. } \angle$,
 $\therefore \angle GBD + \angle GBA = \angle BGE + \angle BGH$. (Axiom.)

But, $\angle GBD = \angle BGE$.

Subtracting; $\angle GBA = \angle BGH$. (Axiom.)

Again, to prove

$\angle DBM = \angle HGB$; $\angle GBA = \angle DBM$. (Why?)
and $\angle GBA = \angle BGH$. (Why?)
 $\therefore \angle DBM = \angle BGH$. (Axiom.)

Again, to prove $\angle DBG + \angle BGH = 2 \text{ right angles}$.
 $\angle ABG + \angle GBD = 2 \text{ right angles}$. (Why?)

But $\angle ABG = \angle BGH$.

$\therefore \angle BGH + \angle DBG = 2 \text{ right angles}$.
(Axiom.)

Proposition XI. (Conversely.)

29. *When a transversal cuts two lines so that the alternate-interior angles are equal, the lines are parallel.*

Statement: Let xy cut AB and FH so that $\angle ACG = \angle CGH$, then AB is $\parallel$ to FH .

Analysis: If a line drawn through C, the point of intersection of AB with xy , parallel to FH and AB, can be identified with this line, our proposition is established. Prop. X will aid the identification.

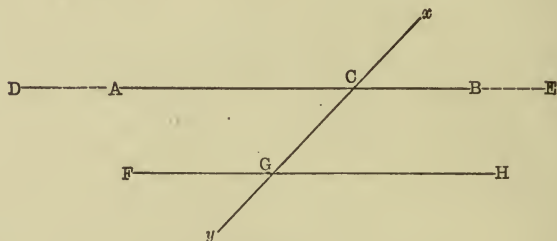


Fig. 15.

Proof: Draw the line DE through C $\parallel$ to FH.

Then $\angle DCG = \angle CGH$. (By Prop. X.)

But $\angle ACG = \angle CGH$. (By hypothesis)

$\therefore \angle DCG = \angle ACG$. (Axiom.)

Since CG is a side common to both angles (DCG and ACG), the other sides (DC and AC) must coincide. $\therefore$ whole line AB coincides with whole line DE. But DE was drawn $\parallel$ to FH. $\therefore$ AB is $\parallel$ to FH.

Triangles.

30. Definitions: A triangle is a portion of a plane inclosed by three straight lines.

31. The bounding lines are called *sides*, and their sum, the perimeter of the triangle. The points of intersection of the lines are called *vertices*. The *adjacent* parts of a triangle (consisting of angles and sides) are those between which no other part intervenes.

32. The angle formed outside a triangle by one of its sides and the prolongation of the adjacent side is called an *exterior angle*: thus, CAB , ECD , and FEA are all exterior angles. The two angles within the triangle *not* adjacent to an exterior angle, are called its *opposite interior* angles.

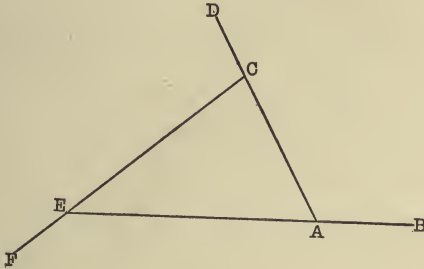


Fig. 16.

33. A triangle is called *scalene* when no two of its sides are equal; *isocles* when two of its sides are equal; *equilateral* when all three sides are equal. A *right triangle* has one right angle. Define an equiangular triangle.

34. In a right triangle, the side opposite the right angle is called the *hypotenuse*, and the other two sides, the *legs*.

35. Any side of a triangle may be called its *base*. It is customary to call the equal sides of an isosceles triangle, the legs and the other side, the base.

36. The angle opposite the side upon which the triangle rests (the base) is called the *vertical* angle.

37. The altitude of a triangle is the *perpendicular* distance from its vertex to its base. The *lines* drawn from the vertices of a triangle to the middle of the opposite sides are called *medians*.

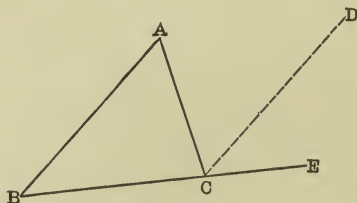
38. The parts similarly situated in different figures are called *homologous* parts.

39. Any two or more figures are said to be *equal* when they can be made to exactly coincide when applied to each other.

Proposition XII.

40. *The sum of the three angles of any plane triangle is equal to two right angles.*

Statement: In the triangle ABC, $\angle ABC + \angle BAC + \angle ACB = 2 \text{ rt. } \angle$.



F.g. 17.

Analysis: All the angles formed on one side of a straight line about one point together make two right angles. If we can prove that the angles of any triangle equals three angles formed about one of its vertices, on the same side of one of its sides (for instance about the vertex C above BC or BC produced), the proposition is proved.

To do this we take advantage of the proposition establishing the equality of angles formed by parallel lines, by drawing the auxiliary line CD $\parallel$ to AB and by producing BC, say to E.

Proof: Since BA and CD are $\parallel$ and are cut by the transversal, a portion of which is the side AC, $\angle BAC =$ the

alternate interior angle $\angle ACD$; also AB and CD are cut by the transversal BE and $\angle ABC =$ its opposite exterior $\angle DCE$.

Then

$$\angle ACB + \angle ACD + \angle DCE = \angle ACB + \angle BAC + \angle ABC.$$

But

$$\angle ACB + \angle ACD + \angle DCE = 2 \text{ rt. } \angle. \quad (\text{Why?})$$

$$\therefore \angle ACB + \angle BAC + \angle ABC = 2 \text{ rt. } \angle. \quad (\text{Why?})$$

41. Cor. I. If two angles of a triangle (or their sum) are known, the third angle may be found by subtracting their sum from two right angles.

42. Cor. II. If two triangles have two angles of one (or their sum) equal to two angles of the other (or their sum) the third angles are equal.

43. Cor. III. If two right triangles have one acute angle in each equal, the other acute angles are equal.

44. Cor. IV. There cannot be two right angles in one triangle. (Why?)

45. Cor. V. In a right triangle the two acute angles are complementary.

46. Cor. VI. An exterior angle of a triangle is equal to the sum of the two opposite interior angles.

Proposition XIII.

47. The difference of two sides of a triangle is less than the third side.

Statement $AC - AB < BC$.

Proof: Since a straight line is the shortest distance between two points $AC < AB + BC$. Subtracting AB from each side leaves $AC - AB < BC$. (Axiom.)

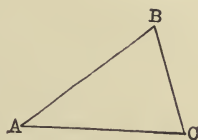


Fig. 18.

Proposition XIV.

48. *Two triangles are equal if two angles and their common side in one, equals two angles and their common side in the other.*

Statement: $\triangle ABC = \triangle DEF$, if $\angle BAC$, $\angle BCA$ and side AC in $\triangle ABC$ equal $\angle EDF$, $\angle EFD$ and side DF in $\triangle DEF$.

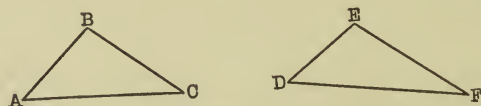


Fig. 19.

Analysis: Since the only test of equality is coincidence, we must apply one triangle to the other and compare. We know that AC can be made to coincide exactly with DF , A being at D .

Proof: Since, $AC = DF$, C will fall exactly on F . Since $\angle BAC = \angle EDF$, AB will take the direction of DE , from definition of angle. For like reason BC will take the direction of EF , and hence they must meet at the same point E .

$\therefore B$ falls on E , and the two triangles exactly coincide, hence they are equal.

49. *Cor I.* Two triangles are equal if a side and any two angles of one are equal to a side and any two angles of other. (Why?)

What parts must be equal in a right triangle, that the triangles may be equal?

Proposition XV.

50. *Two triangles are equal, if two sides and the angle they form in one, are equal to two sides and the angle they form in the other.*

Statement: $\triangle ABC = \triangle DEF$, if
 AB , AC and $\angle BAC$ in $\triangle ABC = DE$, DF and $\angle EDF$
 in $\triangle DEF$.

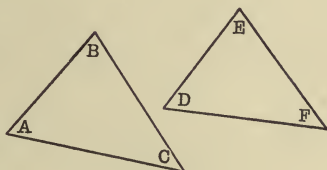


Fig. 20.

Analysis: The final test of equality is coincidence and one triangle must be applied to the other, making one of the equal parts in each coincide at the start.

Proof: Place $\triangle DEF$ upon $\triangle ABC$, making DF fall upon AC . Since $DF = AC$, if D is placed upon A , F will fall upon C . And since $\angle EDF = \angle BAC$, DE will fall upon AB , also DE being equal to AB , E will fall upon B . Since the lines EF and BC have two points each in common, namely B and E , and F and C , EF will coincide with BC .

$\therefore \triangle ABC$ and $\triangle DEF$ coincide throughout, hence are equal.

Proposition XVI.

51. *Two triangles which have three sides of the one equal to three sides of the other are equal.*

Statement: $\triangle ABC = \triangle xyz$ if $AB = xy$, $BC = yz$ and $AC = xz$.

Analysis: Equality is always proved finally by coincidence, but as no angles are given in this case, we cannot determine the relative directions of the sides. We must then resort to some plan that will enable us to find at least one angle.

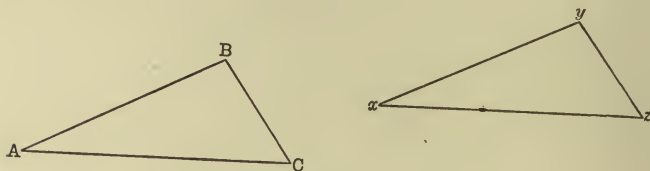


Fig. 21.

Instead of placing one triangle directly over the other, we apply a side of one to its homologous side in the other, making them exactly coincide and then compare the other parts.

Proof: Place the side AC on the side xy , so that A falls on x and C on z , and the vertex B falls on the opposite side of xy from y , thus (Fig. 22.):

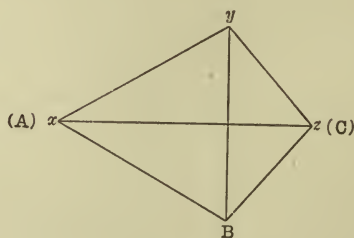


Fig. 22.

Join the vertices y and B .

Now since $xy = AB$ by hypothesis, $xy = xB$ (since x and A are same point now) in the triangle Bxy , and $\angle xyB = \angle xBy$. (Why?)

Likewise yzB is an isosceles triangle and $\angle Byz = \angle yBz$.

But $\angle xyB = \angle xBy$. (By above.)

Add ; $\angle Byz + \angle xyB = \angle yBz + \angle xBy$,

or $\angle xyz = \angle xBz = \angle ABC$.

Then in the two triangles xyz and ABC we have two sides (AB and BC) in one = to two sides (xy and yz) in the other, and the angle ABC between AB and BC in $ABC = \angle xyz$ between xy and yz in xyz , hence the two triangles are equal.

52. *Cor. I.* Two right triangles are equal if their legs are equal.

Proposition XVII.

53. *In an isosceles triangle, the angles opposite the equal sides are equal ; and conversely, if in any triangle two of the angles are equal the triangle is isosceles.*

Statement of first part:

In the isosceles triangle MNO , having $MN = NO$, the $\angle NMO = \angle NOM$.

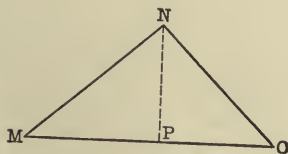


Fig. 23.

Analysis: It is plainly necessary to divide the triangle into two parts to be proved equal; and that can be done only by a line from vertex N .

Proof: Draw NP bisecting $\angle MNO$, then we have two triangles having the sides MN and NO equal, the side NP, common to both, and the angle MNP = angle ONP (halves of same angle), hence the Δ 's are equal. (Why?)

$\therefore \angle NMP = \text{homologous } \angle NOP$.

Statement of second part:

Let $\angle A = \angle C$ in triangle ABC.

To prove $AB = BC$.

Same analysis as first part.

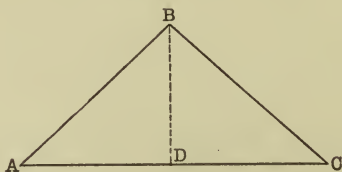


Fig. 24.

Proof: Draw $BD \perp$ to base AC. Then in the two right triangles ABD and DBC, BD is common to both and angle BAD = angle BCD (by hypothesis), hence the triangles are equal. (Why?)

Hence, $AB = BC$ and ABC is isosceles.

54. *Cor. I.* Since AD also equals DC and $\angle ABD = \angle DBC$ (because triangle ABD and BDC are equal), the perpendicular from the vertex of an isosceles triangle to the base bisects the base and also the vertical angle.

55. *Cor. II.* An equiangular triangle is also equilateral.

Proposition XVIII.

56. Two right triangles are equal if a leg and the hypotenuse of one is equal to a leg and the hypotenuse of the other.

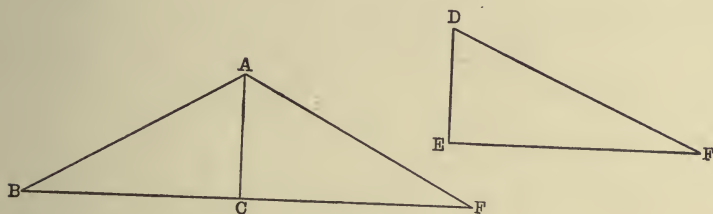


Fig. 25.

Statement: If ABC and DEF have $AB = DF$ and $AC = DE$, then $ABC = DEF$.

Analysis: This is plainly a case for comparison of figures, by some form of superposition.

Proof: Apply DEF to ABC , making the equal sides DE and AC coincide exactly, F and B , being on opposite sides of this common line. Then BC and CF (EF) will form one straight line.

(Why?)

Then the triangle BAF is isosceles because $AB = DF$ (now AF) by hypothesis; hence, $\angle ABF = \angle AFB$, that is, the two right triangles have two sides and an acute angle equal in each, $\therefore$ the triangles are equal.

Proposition XIX.

57. If two sides of a triangle are unequal, the angle opposite the greater side is greater than the angle opposite the smaller side.

Statement: If $BC > AB$ in the triangle ABC , $\angle BAC > \angle BCA$.

Analysis: A comparison of the isosceles triangle formed by laying off a distance on the greater side BC , equal to AB , and joining this point with A , will show the relation between the angles, if we observe that one of the angles of this isosceles triangle is the exterior angle of the remaining triangle.

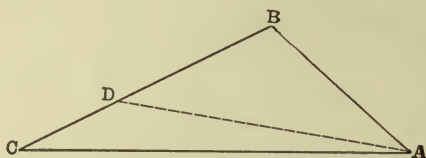


Fig. 26.

Proof: Lay off $BD = AB$ and draw the auxiliary line AD . Then ABD is an isosceles triangle, and hence, $\angle BAD = \angle BDA$. But $\angle BDA$, being the exterior angle of the triangle ADC , is equal to the sum of $\angle DAC$ and $\angle DCA$, hence is greater than either of them, that is $\angle BDA > \angle DCA$, but $\angle BDA = \angle BAD$, which is plainly less than BAC , since it is only a part of $\angle BAC$. That is, since $\angle BDA = \angle BAD$, $\angle BDA < \angle BAC$; but $\angle BDA > \angle DCA$ or $\angle DCA < \angle BDA$, therefore, $\angle DCA$ being less than $\angle BDA$, which is itself less than $\angle BAC$, $\angle DCA$ (or $\angle BCA$) must be less than $\angle BAC$.

Proposition XX.

58. Reciprocally: *If two angles of triangle are unequal, the greater side is opposite the greater angle.*

Statement: If in $\angle ABC$, $\angle ABC$ is greater than ACB , the side $AC >$ side AB .

Analysis and Proof: AB is either equal to, greater than, or less than AC; there is no other alternative.

If $AB = AC$, the triangle is isosceles and angle $ABC =$ angle ACB , which is not true.

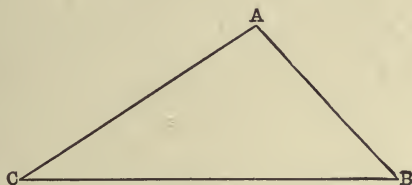


Fig. 27.

If $AB > AC$, by the last proposition $\angle ACB$, opposite AB is greater than $\angle ABC$, opposite AC, but the proposition says $\angle ABC > \angle ACB$, which leaves us but the one alternative, that $AB < AC$.

Proposition XXI.

59. If two triangles have two sides of one equal to two sides of the other, but the angles formed by them unequal, then the third side in the triangle having the greater angle, is the greater.

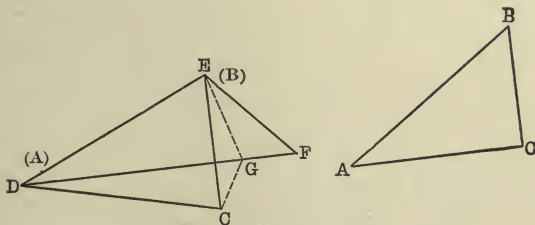


Fig. 28.

Statement: If in triangles ABC and DEF, $AB = DE$, $BC = EF$, and $\angle DEF > \angle ABC$, then $DF > AC$.

Analysis: The two triangles must be compared directly since it is a case of inequality, and hence one must be applied to the other, and auxiliary lines employed to get the relation.

Proof: Apply ABC to DEF making AB coincide with its equal DE; then $\angle ABC$ being less than $\angle DEF$, BC will fall inside of DEF as in the figure.

Draw the auxiliary line EG bisecting $\angle CEF$, meeting DF in G, join C and G.

NOTE. — The wisdom of this construction is apparent, if it be observed that DC and DF are to be compared. Hence if a point on DF can be found (as G) such that if the segment, GF, of DF can be turned about G so that F falls on C, then we have a straight line DC compared with a broken line, which is equal to DF. This point G can only be found by bisecting $\angle CEF$.

In $\triangle$'s EGF and EGC, $EF = EC$ (by hypothesis), EG is common and $\angle GEF$ between EF and EG in $\triangle$ EGF, equals $\angle GEC$ between EG and EC in $\triangle$ EGC, hence $\triangle EGF = \triangle EGC$, and $CG = GF$ (the other sides in the $\triangle$'s EGF and EGC).

In DGC, $DG + GC > DC$. (Why?)

But $GC = GF \therefore DG + GF > DC$,
that is, $DF > DC$, or $DF > AC$.

Proposition XXII.

60. State and prove the converse of Prop. XXI.

Proposition XXIII.

61. *The perpendicular bisector of a line is the locus of points equidistant from the extremities of the line.*

Statement: If xy is the perpendicular bisector of line AC , it is the locus of points equidistant from A and C .

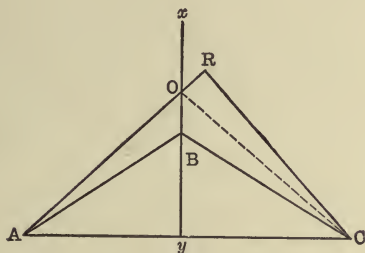


Fig. 29.

Analysis: To show that xy is the specified locus, it is necessary to prove that every point on it is equidistant from A and C , and also that any point outside of xy is unequally distant from A and C .

Proof: Let B be any point on the $\perp$ bisector xy , then in the two right triangles ABy and yBC , By is common and $Ay = yC$ because y is the middle point of AC . Hence triangle $ABy =$ triangle yBC (Why?). Hence $AB = BC$. That is, B (any point on xy) is equally distant from A and C .

2d. Let R be any point outside of xy , then drawing AR , RC , and OC , O being the point where AR cuts xy , $AO = OC$ by first part, and $OR + OC > RC$, that is, since $OC = AO$, $OR + AO > RC$ or $AR > RC$.

Proposition XXIV.

62. The locus of points within an angle and equally distant from the sides is the bisector of the angle.

Statement: Let A be any angle ; then is AM , the bisector, the locus of points within the angle A equally distant from its sides.

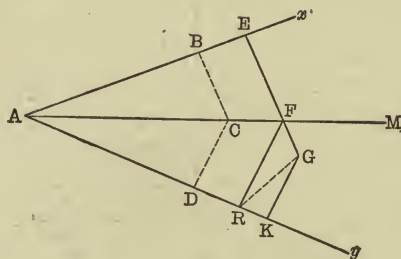


Fig. 30.

Analysis: To prove that AM is the required locus, it is necessary to show that every point on AM is equally distant from Ax and Ay and that any point not on AM is unequally distant; or that every point on AM is equally distant from Ax and Ay and that every point equally distant from Ax and Ay is on AM .

In the first process, then, we must take any point on AM and prove that the right triangles formed by perpendiculars drawn from it to the sides are equal, and taking a point without to show that the perpendiculars drawn from it to the sides are unequal.

Proof: Let C be any point on AM . Draw the perpendiculars CB and CD . Then in the two right triangles ABC and ACD , AC is common and $\angle BAC = \angle CAD$ being halves of same angle, therefore, $\angle ACB = \angle ACD$ (Why?), and we have two angles and the included side, $\therefore$ triangle $ABC =$ triangle ACD , hence side $BC =$ side CD .

2d. Take, G , any point outside of AM , and draw the perpendiculars GE and GK to the sides, then $GK < GE$.

Draw $FR \perp Ay$ from F — the point where GE cuts AM — also RG . (Why?) $\therefore FE + FG > GR$ (since $FE = FR$) or, since $FE + FG = EG$, $EG > GR$. But $GR > GK$. (The perpendicular is shortest distance from a point to a line.) $\therefore EG > GK$.

Exercise: As an exercise prove that every point equally distant from Ax and Ay is on the bisector AM .

QUADRILATERAL.

Definitions.

A *quadrilateral*, as the name signifies, is a four sided figure.

Any side of a quadrilateral may be called its *base*.

The *diagonal* of any quadrilateral is the straight line joining any two vertices not adjacent.

The *vertices* are the points where the sides intersect or meet.

A *parallelogram* is a quadrilateral whose opposite sides are parallel. Draw one.

A *trapezoid* is a quadrilateral two of whose opposite sides only are parallel. Draw one.

A *trapezium* is a quadrilateral no two sides of which are parallel. Draw one.

The *altitude* of a parallelogram is the distance between either pair of parallel sides; of a trapezoid, it is the distance between its parallel sides. Draw the altitude of each.

A *rectangle* is a parallelogram whose angles are all right angles. A rectangle with all sides equal is called a *square*. Draw each.

A *rhomboid* is a parallelogram with oblique angles.

A *rhombus* is an equilateral rhomboid. Draw one.

Proposition XXV.

63. *The diagonal of a parallelogram divides it into two equivalent triangles.*

Statement: If $ABCD$ is a $\square$, BD is a diagonal and divides the $\square$ into the two equivalent triangles ABD and DBC .

Analysis: Triangles are equal if they have two sides and included angle; or two angles and included side, or three sides equal each to each.

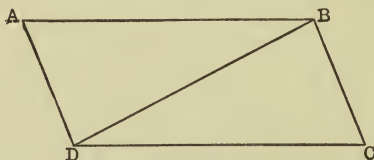


Fig. 31.

We may choose any one of these methods of comparison that suits our data.

Proof: BD is common to both triangles, and since, by definition, AD is $\parallel$ to BC and AB is $\parallel$ to DC , and each of these parallel pairs is cut by the transversal DB , $\angle ABD = \angle BDC$ and $\angle ADB = \angle DBC$. What follows?

64. Cor. Since the triangles $ABD =$ triangle BDC , their homologous parts are all equal, hence the parallel sides of a parallelogram are also equal.

NOTE. — Two figures are said to be *equivalent* if they inclose equal space, but cannot be made to coincide; *equal*, if they inclose equal space and will also coincide.

Proposition XXVI.

65. *The two diagonals of a parallelogram bisect each other.*

Statement: ABCD is a $\square$; then its diagonals AC and DB bisect each other, that is, E is the middle point of both.

Analysis: Same as Prop. XXV.

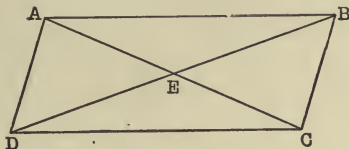


Fig. 32.

Proof: In the triangle AEB and DEC, AB is equal and parallel to DC by last proposition. They are cut by the transversals AC and DB.

Complete the proof.

Exercise: State and prove the converse of above proposition.

Proposition XXVII.

66. If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

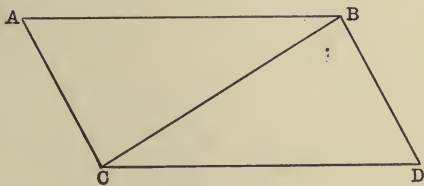


Fig. 33.

Statement: If AB and CD are equal and parallel, then ABDC is a parallelogram.

Analysis: By definition a parallelogram has parallel opposite sides; also two lines are parallel if they do not

meet, or if a transversal makes equal alternate-interior angles with them. It is clear that the latter criterion must be used here. It remains then to prove, for instance, that the diagonal makes equal angles with AC and BD, which can be most easily established by proving the equality of the triangles containing them.

Proof: Draw a diagonal CB; then in the two triangles ABC and BCD, AB equals CD, and since also AB is $\parallel$ to CD, and both are cut by CB, the alternate angles ABC and BCD are equal; also CB is common, $\therefore$ triangle ABC equals triangle CBD. (Why?)

Hence, $\angle BCA$ equals $\angle CBD$, and $\therefore$ AC is $\parallel$ BD.

Since the sides are parallel in pairs, ABDC is a parallelogram.

Exercise: Prove that, if the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

Proposition XXVIII.

67. *If three or more parallels intercept equal parts of one transversal, they intercept equal parts of every transversal.*

Statement: If the parallels GF, DK, CL, BM, etc., intercept equal parts of EM as $FK = KL = LM$, etc., then they intercept equal parts on any other line AB, as $DG = DC = CB$, etc.

Analysis: To compare lines or geometrical magnitudes of any sort it is, as a general rule, necessary to connect them or their equivalents. If we draw from G, D, and C, lines $\parallel$ to EM, we evidently get the equivalents of FK, KL, LM, etc., and since these $\parallel$'s are connected with DG, CD, BC, etc., we ought to be able to derive relations between them.

Proof: Draw GX, DY, CZ, etc., $\parallel$ to EM.

In the triangles DGX and DYC, $GX = DY$ (for $GX = FK$ and $DY = KL$ (Why?), and $FK = KL$ by hypothesis),

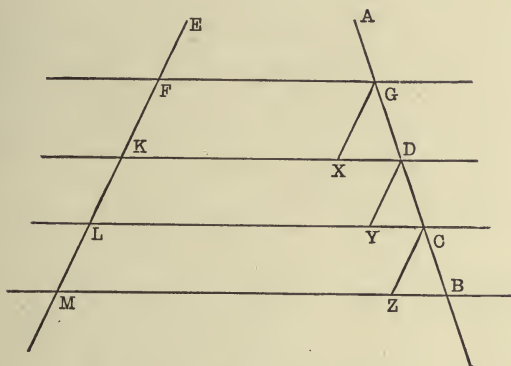


Fig. 34.

$\angle DGX = \angle CDY$ (since GX is $\parallel$ to DY and they are cut by AB), $\angle GDX = \angle DCY$ (since DK is $\parallel$ to CL and cut by AB), hence $\angle GXD = \angle DYC$. (Why?)

Then we have two angles and the included side, $\therefore$ triangle $DGX =$ triangle CDY , and hence $DG = CD$; likewise, CD can be proved equal to BC , etc.

68. Cor. I. If a line is parallel to one side of a triangle; and bisects another side, it bisects the third side also. That is, in the triangle AEB , if DC is $\parallel$ to AB and bisects AE it bisects EB also.

Draw an auxiliary line through E $\parallel$ to DC and AB and apply Prop. XXVIII. Complete proof (Fig. 35).

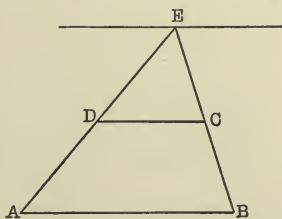


Fig. 35.

69. *Cor. II.* The line joining the mid-points of two sides of a triangle is parallel to the third side and is one half the length of the third side (Fig. 36).

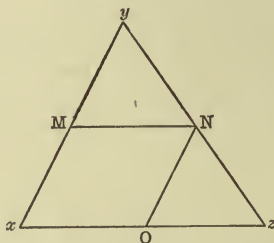


Fig. 36.

That is, if M and N are the mid-points, respectively, of xy and zy , then MN is $\parallel$ to xz and $MN = \frac{1}{2} xz$.

First part: A line drawn through N $\parallel$ to xz passes through M by Cor. I, and since two points determine a straight line, MN must be this line which is parallel to xz .

Second part: Draw ON through N $\parallel$ to xy . Then, since MN is $\parallel$ to xO and ON is $\parallel$ to xM , $xMNO$ is a parallelogram; hence $MN = xO$, but O is the mid-point of xz . (Why?)

$$\therefore MN = xO = \frac{1}{2} xz.$$

70. *Cor. III.* The median of a trapezoid is parallel to the bases, and is equal to half the sum of the bases.

That is, DC is $\parallel$ to AB and $FG = \frac{1}{2} (AB + FG)$.

First part: Draw the diagonal AG and let E be its mid-point.

Connect E with D and C by straight lines DE and EC ; since AB is $\parallel$ to FG , DE and EC are both $\parallel$ to AB and FG , and since only one line could be drawn through the point

$E \parallel$ to either AB or FG , BE and EC must be collinear, that is, form parts of the same straight line.

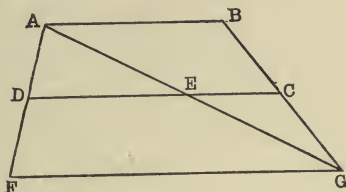


Fig. 37.

Second part: In the triangles AFG and ABG , by Cor. II,

$$DE = \frac{1}{2} FG,$$

and

$$EC = \frac{1}{2} AB.$$

Add ;

$$DE + EC = \frac{1}{2} (FG + AB),$$

or

$$DC = \frac{1}{2} (FG + AB).$$

POLYGONS.

Definitions.

71. A *polygon* is a plane figure of any number of sides.

The *vertices* of a polygon are the points where the sides intersect.

A *diagonal* of a polygon is a line joining any two vertices not adjacent.

A polygon is said to be *equilateral* where its sides are all equal; *equangular*, when its angles are all equal.

A polygon is said to be *convex* when no side produced will enter the polygon; as, 

If any side producing does enter the polygon it is said to be *concave*; as 

The angle within a concave polygon, formed by the sides which would enter the polygon if produced, is called a re-entrant angle.

Only convex polygons are considered here. The mutually equal angles in two or more equangular polygons are called homologous angles, and the sides including these equal angles are called homologous sides.

A polygon of three sides is a triangle.

A polygon of four sides is a quadrilateral.

A polygon of five sides is a pentagon.

A polygon of six sides is a hexagon.

A polygon of seven sides is a heptagon.

A polygon of eight sides is an octagon, etc.

This may be represented schematically as follows:

polygons.	{	triangle	3 sides.
		quadrilateral	4 sides.
		pentagon	5 sides.
		hexagon	6 sides.
		heptagon	7 sides.
		octagon	8 sides.
		nonagon	9 sides.
		decagon	10 sides, etc.

Proposition XXIX.

73. *The sum of the interior angles of a polygon is equal to two right angles taken as many times less two, as the polygon has sides.*

Statement: In the polygon ABCDEF, $\angle A + \angle B + \angle C + \angle D + \angle E + \angle F =$ two right angles taken $(6 - 2)$ times, that is $=$ eight right angles.

Analysis: Since we can divide the polygon into triangles by diagonals from any vertex, and since we know the sum

of the angles in any triangle, it is desirable to make this division.

Proof: Draw diagonals from A to C, D and E, forming triangles BAC, CAD, DAE, EAF.

Since each of the outside triangles ABC and AFE use two of the sides of the polygon, and each of the other tri-

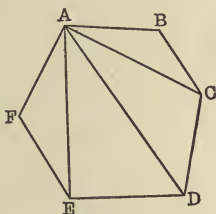


Fig. 38.

angles one, there will be two less triangles than sides of the polygon.

The sum of the angles in all of the triangles is equal to the sum of the angles of the polygon.

There being two less triangles than sides of the polygon, there will be two right angles as many times as the polygon has sides, less two.

In general, if the polygon has N sides, there will be two right angles repeated $N - 2$ times.

Proposition XXX.

74. *The sum of the exterior angles of a polygon equals four right angles.*

Statement: If the sides of the polygon ABCFHE are produced in order, the sum of the exterior angles ABE, BCD, CFG, FHK, LAH = four right angles.

Analysis: The exterior angles and their corresponding interior angles at each vertex equal two right angles and we know the sum of the interior angles.

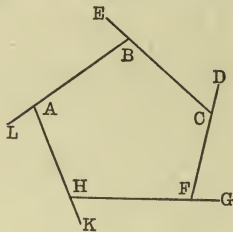


Fig. 39.

Proof: As an illustration, a pentagon is taken, but what is true of the pentagon is evidently true of any other polygon, as to angle-sums,

$$EBA + ABC = \text{two right angles.}$$

$$BCD + BCF = \text{two right angles.}$$

$$CFG + CFH = \text{two right angles.}$$

$$FHK + FHA = \text{two right angles.}$$

$$HAL + HAB = \text{two right angles.}$$

Add; $(EBA + BCD + CFG + FHK + HAL) + (ABC + BCF + CFH + FHA + HAB) = 10 \text{ rt. angles.}$

But by the last proposition,

$$ABC + BCF + CFH + FHL + HAB = 2(5 - 2) = 6 \text{ rt. angles.}$$

$$\therefore EBA + BCD + CFG + FHK + HAL = 4 \text{ right angles} = (10 - 6) \text{ right angles.}$$

In general, if there are n sides, the sum of the exterior angles + sum of the interior angles = $n \times 2 \text{ rt. angles}$ and sum of the interior angles = $(n - 2) 2 \text{ rt. } \angle\text{s} = (2n - 4) \text{ rt. } \angle\text{s}.$

$$\therefore \text{Sum of exterior angles} + (2n - 4) \text{ rt. } \angle\text{s} = 2n \text{ rt. } \angle\text{s}.$$

$$\therefore \text{Sum of exterior angles} = [2n - (2n - 4)] \text{ rt. angles} = 4 \text{ rt. angles.}$$

EXERCISE I.

1. Show that the bisector of an angle, also bisects its vertical angle.
2. Show that the bisectors of two supplementary adjacent angles are perpendicular to each other.
3. One of the four angles, formed by the intersection of two lines, equal $\frac{3}{4}$ of a right angle. Find each of the other 3.
4. The sum of two of the vertical angles formed by the intersection of two lines, is 3 times the sum of the other two. What is the value of each angle?
5. An angle is 5 times its supplementary adjacent angle. What is its value?
6. Two parallel lines are cut by a transversal; prove that the alternate exterior angles are equal.
7. Prove that the bisectors of the two alternate interior angles of two parallels cut by a transversal, are parallel.
8. The bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base.
9. Find the angles A, B, and C of a triangle if $A = 2B$ and $B = 2C$.
10. If the vertical angle of an isosceles triangle is $\frac{2}{3}$ of a right angle, show that the triangle is equilateral.
11. Show that the sum of the lines drawn from any point inside a triangle to the vertices is greater than half the perimeter and less than the whole perimeter.
12. Can sides of the length 7, 8, and 17 inches respectively form a triangle. Give reason for answer.
13. In the isosceles triangle xyz , A being a point in xz a line is drawn from y to A so that $xA > Az$. Prove $\angle zAy < yAz$.
14. In an isosceles triangle show that the bisectors of the base angles, form an isosceles triangle with the base.

15. If one angle of a parallelogram is $\frac{2}{3}$ of a right angle, what are the other angles?

16. If two sides of a quadrilateral are equal and two of the respectively opposite angles are equal, it is a parallelogram.

17. Any line drawn through the intersection point of the diagonals of a parallelogram and terminating in opposite sides is bisected by the point.

18. In an isosceles trapezoid the angles at the extremities of a base are equal.

19. The line joining two points on opposite sides of a given line and equally distant from it, is bisected by the given line.

20. What is the size of each interior angle of a regular decagon?

21. One interior angle of a regular polygon is $1\frac{3}{4}$ right angles, how many sides has the polygon?

22. In laying a tile floor, what different kinds of regular polygons can be used, and how many of each kind will be required around any one point?

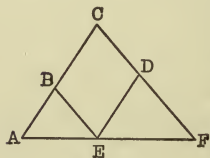


Fig. 39a.

23. The lines joining the middle points of the opposite sides of a quadrilateral bisect each other.

24. If from any point in the base of an isosceles triangle (as E, in ACF) two lines be drawn parallel respectively to the equal sides, the perimeter of the quadrilateral (BEDC) thus formed equals the sum of the equal sides (AC and CF) and hence is constant.

25. How many sides has a polygon, if the sum of its interior angles exceeds the sum of its exterior angles by 10 right angles?

26. The middle point of the hypotenuse of a right triangle is equally distant from the three vertices.

27. Show how to determine a point in a straight line, the sum of whose distances from two fixed points on the same side of the line (and unequally distant from it) shall be the least possible.

28. Show that the lines in 27 joining the two exterior points with the point in the line, make equal angles with the line.

29. If a line, drawn through the vertex of a triangle, parallel to its base, bisects the exterior angle, the triangle is isosceles.

30. The angle, formed by the bisectors of any two angles of an equilateral triangle, is double the other angle.

31. If the diagonals of a parallelogram are perpendicular to each other and equal, the figure is a square.

32. The angle formed by the bisectors of the interior angle between the base and one side of a triangle, and of the exterior angle between the base produced and the other side, is one half the vertical angle of the triangle.

THE CIRCLE.

Definitions.

74. A *circle* is a plane figure whose boundary is a curved line, every point of which is equally distant from a point within, called its center.

The boundary curve is called the *circumference*.

75. The *radius* is a straight line joining the center to any point on the circumference.

The word *radius* often refers to the length of this line, not necessarily to the line itself.

76. The *diameter* is a straight line through the centre joining any two opposite points on the circumference.

Necessarily all radii and all diameters of the same circle are equal, and the diameter is twice the radius.

77. An *arc* is any part of the circumference.

A *chord* is a straight line joining the ends of an arc.

A chord is said to *subtend* its arc.

Every chord subtends two arcs from the nature of a circle. The smaller one is the one usually referred to.

78. A polygon is said to be *inscribed* in a circle when its vertices are points in the circumference. The circle is said to be *circumscribed* about the polygon.

79. An angle is said to be inscribed in a circle when its vertex is in the circumference and its sides are chords.

80. A *segment* of a circle is the portion of it bounded by any arc and its chord.

81. A *sector* of a circle is the portion of it bounded by any two radii and the arc between their extremities.

Represent a circle with all the parts defined.

82. Clearly any diameter divides a circle into two equal parts as also its circumference.

Also circles having equal radii are equal.

Proposition I.

83. *A diameter is the greatest chord of any circle.*

Statement: Let AB be a diameter of the circle O, then is AB greater than any other chord, say CD.

Analysis: Since all diameters are equal, we can draw another diameter through either end of the chord and thus get a comparison, remembering that a diameter is twice a radius.

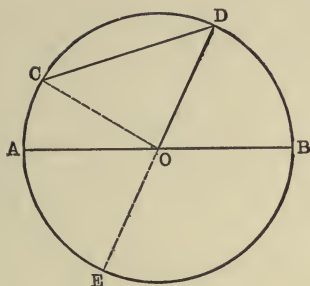


Fig. 40.

Proof: Draw another diameter DE through D; also join C and O. Then $DO + CO > CD$, but $DO + CO = DE$ (Why?) and $DE = AB$, $\therefore AB > CD$, any other chord.

Proposition II.

84. *In the same circle or in equal circles, equal angles at the centre intercept equal parts of the circumference (equal arcs).*

Statement: Let A and B be two equal circles and $\angle DAC = \angle FBG$, then arc DC = arc FG.

Analysis: Superposition.

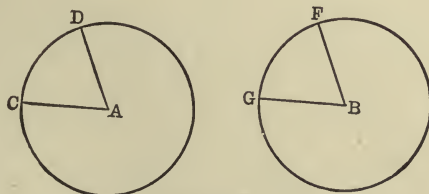


Fig. 41.

Proof: Place circle B on circle A, making BG coincide with AC; since the radii are equal G falls on C, and since

$\angle DAC = \angle GBF$, BF falls on AD , and $BF = AD$, $\therefore F$ falls on D , hence arc FG coincides with arc DC . (Why?)

Prove the converse of Prop. II.

Definitions.

85. The last proposition would suggest that an arc would make a convenient measure for an angle. Since four right angles is the aggregate of all angles formed about a point, no more, no less, the sum of angles formed at the centre of every circle is the same, hence every circumference, if used as a measure, must contain the same number of units.

It has been agreed that every circumference shall be said to contain 360 of these units, called *degrees*. As the circles vary in size, plainly the lengths of these degrees vary with the circle; hence it is clearly necessary, in comparing angles, to employ parts of the same circumference or of circumferences having equal radii.

86. Each degree is arbitrarily divided into 60 *minutes*, and each minute into 60 *seconds*. Compare the method of determining units of linear measure.

87. A *secant* line is any line which cuts the circle; or a line having two points in common with the circle.

88. A *tangent* is a line touching the circle in only *one* point; or a line having one point in common with the circle.

Proposition III.

89. *In the same circle or equal circles, equal arcs are subtended by equal chords; and of two unequal arcs the greater is subtended by the greater chord.*

Statement: If M and N are two equal circles, and the arcs

AB and OR are equal, then chord AB = chord OR; and if arc $SR > \text{arc } AB$, chord $SR > \text{chord } AB$.

Analysis: Since these chords can easily be incorporated into triangles their comparison is simple.

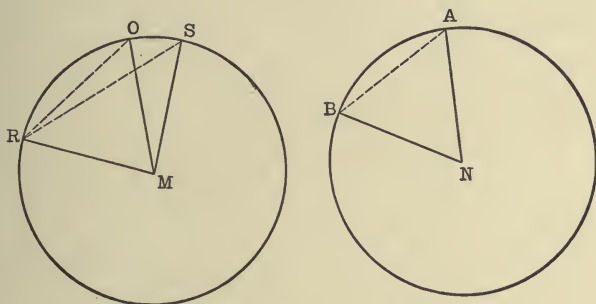


Fig. 42.

Proof: Draw radii MR, MO, and MS, also chords OR and OS in M, and NB and NA, also chord AB in N. In triangles MOR and NAB,

$$MO = NA$$

$$MR = NB.$$

$\angle OMR = \angle ANB$ (Why?), $\therefore$ triangle MOR equals triangle NAB and $\therefore$ chord OR = chord AB.

Again in triangles RMS and OMR, SM = OM and MR is common but $\angle RMS > \angle OMR$ (by hypothesis),

$$\therefore SR > OR \quad (\text{Why?})$$

or $SR > AB = OR.$

EXERCISE.

State and prove the converse of Prop. III.

Proposition IV.

90. A diameter perpendicular to a chord bisects the chord and the arc it subtends.

Statement: Let A be any circle and BD a chord in it, then the diameter FE (or the radius AE) $\perp$ to BD bisects BD at C and also the arc BED at E.



Fig. 43.

Analysis: The proof that C is the middle point of BD, that is, $BC = CD$, is evidently dependent upon the equality of the right triangles BAC and CAD.

Proof: Draw the radii AB and AD, forming with AC and BD two right triangles, BAC and CAD, wherein $AB = AD$ and AC is common, $\therefore BC = CD$ and also $\angle BAC = \angle CAD$, hence arc BE = arc ED.

What might be inferred as corollaries?

Proposition V.

91. In the same circle (or in equal circles) equal chords are equally distant from the centre.

Statement: In the circle M, the chord OS = chord PR, then $NM = MQ$.

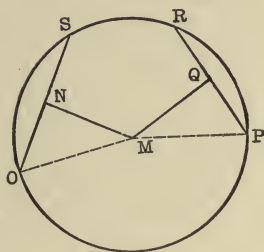


Fig. 44.

Analysis: Form right triangles with radii.

Proof: Draw radii MO and MP. Then in the right triangles MON and PMQ, OM = MP, and ON = PQ (Why?), $\therefore$ MN = MQ.

State and prove the converse of Prop. V.

Proposition VI.

92. *In the same circle (or in equal circles) unequal chords are unequally distant from the centre and the greater chord is at the lesser distance.*

Statement: If in the circle D, the chord EG > chord AC, BD > FD.

Analysis: If we can include these lines BD and FD in the same triangle, they can readily be compared. By Prop. V, the chord EG and its distance from the centre FD can be duplicated by a chord drawn from C or A, and hence a connection established with AC.

Proof: Draw CK = EG, also DL perpendicular to CK. Then by Prop. V, DL = FD connect B and L. Then in the triangle BCL, thus formed, CL > BC, for $CL = \frac{1}{2} CK$ and $BC = \frac{1}{2} AC$ (Why?) and CK > AC,

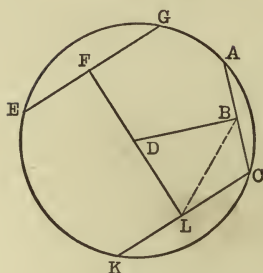


Fig. 45.

hence $\angle CBL > \angle CLB$ (Why?). $\angle CBD = \angle CLD$, (both right angles).

Subtract $\angle CBL > \angle CLB$.
 $\angle CBD - \angle CBL = \angle LBD < \angle BLD = \angle CLD - \angle CLB$,
 since the greater of two quantities leaves the less remainder when they are subtracted from the same or equal quantities.

But if $\angle LBD < \angle BLD$, $DL < BD$ (why?), or since $DL = DF$, $DF < BD$.

EXERCISE.

State the converse of Prop. VI, and indicate the method of proof.

Proposition VII.

93. *A tangent to a circle is perpendicular to the radius drawn to the point of tangency.*

Statement: If FE is tangent to the circle A at B, then it is perpendicular to AB.

Analysis: The perpendicular is the shortest line from a point to a line.

Proof: Draw AD any other line from A to FE, cutting the circle at C. Since by definition the tangent to a circle touches it in but one point the point D must lie outside the circle, hence $AD > AC$, but $AC = AB$.

$\therefore AD > AB$; hence AB is shorter than any other line from A to FE. $\therefore$ AB is perpendicular to FE.

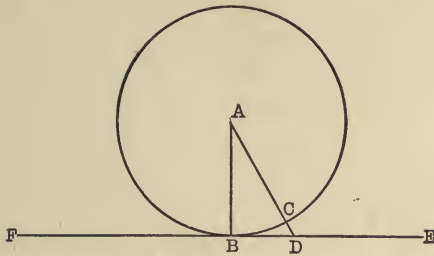


Fig. 46.

94. *Cor. I.* A straight line perpendicular to a radius at its extremity, is tangent to the circle.

95. *Cor. II.* A perpendicular to a tangent at the point of contact, passes through the centre of the circle.

Proposition VIII. Construction.

96. *To draw a circumference through any three points not in the same straight line.*

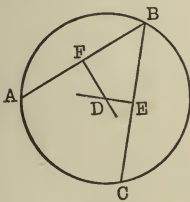


Fig. 47.

Statement: Let A, B and C be any three points not in the same straight line; to pass a circumference through them.

Analysis: Since every point of a circumference must be equally distant from its centre, a point must be found equally distant from A, B and C, which

immediately suggests the perpendicular bisector.

Proof: Join A, B and C by the straight lines AB and BC. Erect to them the perpendicular bisectors DF and DE respectively. Now DF, being the perpendicular bisector of AB, contains all points equally distant from A and B, and DE, for a like reason, contains all points equally distant from B and C; hence the point where DF and DE meet (why do they meet?) will then be equally distant from A, B

and C, because it is a point common to both perpendicular bisectors and possesses all the characteristics of both. Hence with the point of intersection, D, as a centre and a radius DB, a circumference will pass through A, B and C.

It is the only circumference that can pass through A, B and C because two lines (the perpendicular bisectors in this case) can intersect in only one point.

97. Cor. Two circumferences cannot intersect in more than two points. (Why?)

Proposition IX.

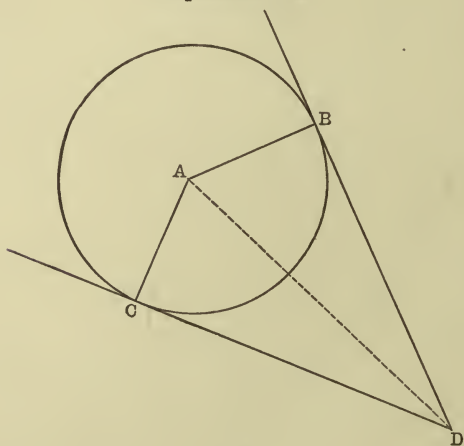


Fig. 48.

98. *The tangents to a circle from an external point are equal in length, and make equal angles, with the line joining the point to the centre.*

Statement: The tangents DB and DC drawn from the point D to the circle A are equal and $\angle BDA = \angle ADC$.

Analysis: Comparison of two right triangles. Complete the proof.

Proposition X.

99. *If two circles intersect, the line of centers is perpendicular to their common chord at its middle point.*

Statement: Let the circles M and N intersect, with common chord OP, then MN is $\perp$ to OP at its middle point O'.

Analysis: The conditions for perpendicular bisector must be established.

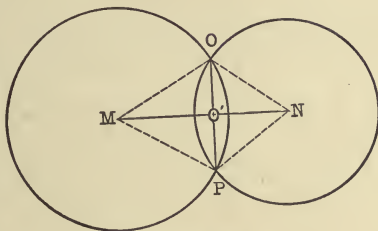


Fig. 49.

Proof: Join O and P with both M and N. Then $ON = NP$ and $OM = MP$, hence M and N are each equally distant from O and P, hence they are points on perpendicular bisector.

100. Cor. If two circles are tangent, the line of centers passes through the point of contact. (Why?)

Proposition XI. Construction.

101. *To find the ratio of two given lines.*

Let the lines be AB and CD.

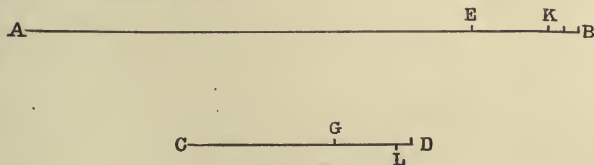


Fig. 50.

Lay off CD on AB as many times as possible; in this case, twice with a remainder EB.

Lay off EB on CD as many times as possible; here twice, with a remainder GD.

Lay off GD on EB as many times as possible; here once, with remainder KB.

Lay off KB on GD as many times as possible; here twice, with remainder LD.

LD goes into KB exactly twice, say.

Then

$$KB = 2 LD \quad (1).$$

$$GD = GL + LD = 2 KB + LD = 5 LD \quad (\text{from (1)}) \quad (2)$$

$$EB = EK + KB = GD + KB = 7 LD \quad \dots \quad (3)$$

from ((1) and (2))

$$CD = CG + GD = 2 EB + GD = 19 LD \quad \dots \quad (4)$$

from ((2) and (3))

$$AB = AE + EB = 2 CD + EB = 45 LD \quad \dots \quad (5)$$

from ((3) and (4))

$$\therefore \frac{CD}{AB} = \frac{19 LD}{45 LD} = \frac{19}{45}.$$

By the same process, plainly, any two lines, straight, or of the same curvature, may be compared.

Proposition XII.

102. *In the same circle or in equal circles, two central angles have the same ratio as their intercepted arcs.*

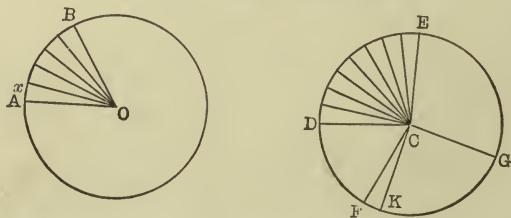


Fig. 51.

Statement: Let O and C be equal circles having central angles BOA and ECD commensurable; also BOA and FCG incommensurable.

$$\text{Then } \frac{\angle BOA}{\angle ECD} = \frac{\text{arc } AB}{\text{arc } ED} \text{ and } \frac{\angle BOA}{\angle FCG} = \frac{\text{arc } AB}{\text{arc } FG}.$$

Analysis: A case for common measure, clearly.

Proof: *Case I.* Arcs AB and ED commensurable. Take any common measure, as small arc x , with corresponding central angle, in an equal circle.

Say it is contained in AB , m times, and in ED , n times, then

$$\frac{\text{arc } AB}{\text{arc } ED} = \frac{m}{n}.$$

Draw radii in each circle to the points of division, dividing the angle AOB into m parts, and the angle ECD into n parts, all equal. (Why?)

$$\text{hence, } \frac{\angle AOB}{\angle ECD} = \frac{m}{n}$$

combining these two proportions.

$$\frac{\angle AOB}{\angle ECD} = \frac{\text{arc } AB}{\text{arc } ED}.$$

Case II. Arcs AB and FG , incommensurable. Apply the same unit of measure x to ACB and FCG ; say, it is contained in AOB m times (as before) and leaves a remainder in FCG , represented by KCF .

Now the arcs AB and KG are commensurable, for both contain the common measure, hence by Case I,

$$\frac{\angle AOB}{\angle KCG} = \frac{\text{arc } AB}{\text{arc } KG}.$$

In division the remainder must always be less than the divisor, hence the remainder-arc KF is less than the unit used.

If a smaller unit is employed to compare the two arcs (and angles), there will be a smaller remainder.

Suppose a unit is chosen (contained exactly in AB) so small that the remainder is too small to be even imagined; although the point K representing the initial point of the remainder — arc FK is so near F that this remainder-arc is inconceivably small, still the proportion

$$\frac{\angle AOB}{\angle KCG} = \frac{\text{arc } AB}{\text{arc } KG}$$

holds true, because AB and KG both contain this very small unit an integral number of times. Since by decreasing the unit (which we can do at will) we can bring the point K as near to F as we please, and the above proportion will continue always to be true, it will hold also at F .

$$\therefore \frac{\angle AOB}{\angle FCG} = \frac{\text{arc } AB}{\text{arc } FG}.$$

Proposition XIII.

103. *An inscribed angle is measured by half the arc intercepted by it.*

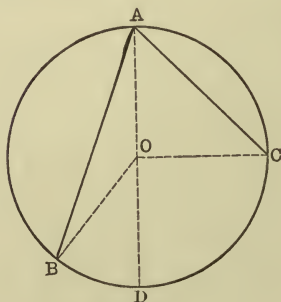


Fig. 52 a.

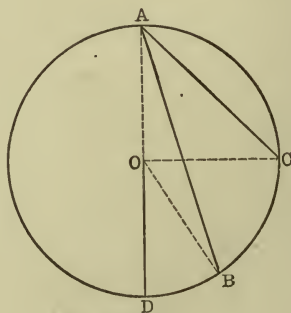


Fig. 52 b.

Statement: Let BAC be an angle inscribed in circle O , intercepting arc BC , to prove $\angle BAC = \frac{1}{2} \text{ arc } BC$.

Analysis: Since we know that a central angle is measured by its arc, it is necessary to express this inscribed angle in terms of a central angle (or central angles) or of its arc.

Proof: Draw the diameter AD from the vertex A, also the radii OC and OB to points C and B. Triangles AOB and AOC are isosceles. (Why?)

$$\therefore \angle OAB = \angle OBA \text{ and } \angle OAC = \angle OCA.$$

Also $\angle DOB$ is the exterior angle of the triangle OBA. hence, $\angle DOB = \angle OAB + \angle OBA = 2 \angle OAB$.

For same reason $\angle DOC = \angle OAC + \angle OCA = 2 \angle OAC$.

That is, $\angle OAC = \frac{1}{2} \angle DOC$
and $\angle OAB = \frac{1}{2} \angle DOB$

$$\angle BAC = \frac{1}{2} \angle BOC$$

subtracting if diameter AD is outside the angle; adding if it is within. (See Fig. 52 *b* and Fig. 52 *a*.)

But $\frac{1}{2} \angle BOC = \frac{1}{2} \text{ arc BC.}$ (Why?)

$$\therefore \angle BAC = \frac{1}{2} \text{ arc BC.}$$

104. Cor. I. An angle inscribed in a semicircle is a right angle.

105. Cor. II. An angle is acute or obtuse according as it is inscribed in a segment greater or less than a semi-circle.

Proposition XIV.

106. An angle formed by the intersection of two chords of a circle is measured by half the sum of the intercepted arcs.

Statement: If AC and BD, two chords of the circle O, intersect at E, the $\angle BEC$ (or its equal AED) is measured by $\frac{1}{2} (AD + BC)$; and $\angle AEB$ (or its equal DEC) is measured by $\frac{1}{2} (AB + CD)$.

Analysis: Since we have learned the relation between central angles and their arcs; also between inscribed angles

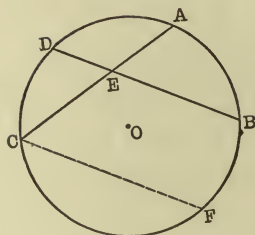


Fig. 53.

and their arcs, it is clearly desirable to express the angles under discussion in terms of either an inscribed or of a central angle. To that end a line drawn from either extremity of either chord, parallel to the other, will form a related inscribed angle.

Proof: From C draw the chord $CF \parallel$ to BD , then by Prop. XIII $\angle ACF$ is measured by $\frac{1}{2}$ arc ABF, and $\angle ACF = \angle AEB$ (Why?). $\therefore \angle AEB = \frac{1}{2}$ arc ABF, but arc ABF = arc AB + arc BF = arc AB + arc CD (arc BF = arc CD, being between parallel chords), hence $\angle AEB = \frac{1}{2}(\text{arc AB} + \text{arc CD})$.

Proposition XV.

107. *An angle, included by a tangent and a chord drawn from the point of contact, is measured by half the intercepted arc.*

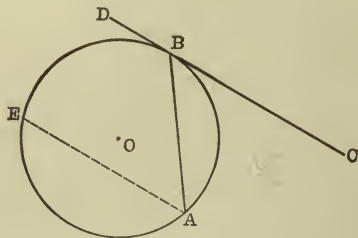


Fig. 54.

Statement: Let CD be a tangent to the circle O, meeting the chord AB at B, the point of tangency, then $\angle ABC$ is measured by $\frac{1}{2}$ arc AB.

Analysis: An inscribed angle is again the means of comparison.

Proof: Draw AE $\parallel$ to CD, then $\angle EAB = \frac{1}{2} \text{EB}$ ($= \frac{1}{2} \text{AB}$) by Prop. XIII, and $\angle EAB = \angle ABC$. $\therefore \angle ABC = \frac{1}{2} \text{AB}$.

Proposition XVI.

108. *An angle formed by two secants, two tangents, or a tangent and a secant, drawn to a circle from an exterior point, is measured by half the difference of the intercepted arcs. The analysis is the same as that for Props. XIV and XV. Prove the Proposition.*

EXERCISE II.

1. A diameter of a circle and a chord are produced outside the circle to meet. Show that the part of the diameter lying outside is less than the part of the chord outside.
2. If the circumference of a circle is divided into four equal parts, the chords joining these points form a rhombus or a square.
3. If a line joining the middle points of two chords is a part of a diameter, the chords are parallel.
4. A chord of the larger of two unequal concentric circles, that is tangent to the inner circle, is bisected at the point of tangency.
5. The shortest chord that can be drawn in a circle through a given point in it is perpendicular to line joining that point and the centre.

6. Two circles are tangent internally, and chords of the larger circle are drawn from the point of tangency through the extremities of a diameter of the smaller circle. Show that the line joining the ends of these chords is a diameter of the larger circle.

7. The circumference of a circle is divided into 5 parts, such that one part is twice the second part, 3 times the third, 4 times the fourth, and 5 times the fifth. What is the length (in degrees) of each arc?

8. If two circles are tangent, the tangents drawn at the ends of two secants, through the contact point, are parallel.

9. Three consecutive sides of an inscribed quadrilateral subtend arcs of 82° , 99° , 67° respectively. Find each angle of the quadrilateral in degrees.

10. If two chords intersect at right angles within a circle, the sum of the opposite arcs intercepted equals a semi-circumference.

11. Two intersecting chords making equal angles with the diameter through their point of intersection are equal.

12. The sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides.

13. The straight line joining the middle of the two non-parallel sides of a circumscribed trapezoid equals $\frac{1}{4}$ its perimeter.

14. The bisector of the angle formed by two tangents to a circle passes through the centre.

15. The circle described on one of the legs of an isosceles triangle as diameter bisects the base.

16. If a perpendicular be dropped from the centre of the circumscribed circle upon the base of a triangle, the angle

between this perpendicular and a radius to the extremity of the base equals the vertical angle.

17. What is the locus of the middle points of a system of equal chords.

18. If a straight line be drawn through the point of contact of two circles, tangent externally, ending in the circumferences, the radii drawn to its extremities are parallel.

19. The sum of the angles subtended at the centre of a circle by two opposite sides of a circumscribed quadrilateral is equal to two right angles.

20. If a circle be inscribed in a right triangle the sum of its diameter and the hypotenuse is equal to the sum of the legs.

21. If two circles intersect and diameters be drawn in each circle from one of the intersection points, a line joining the other ends of these diameters passes through the second point of intersection.

22. The angle subtended by a railroad curve at the centre is 18° . What is the angle between the tangent at one extremity and the chord?

23. What is the value of each angle of a regular inscribed polygon of 24 sides?

24. What is the size of an angle of a regular circumscribed polygon of 15 sides?

25. If two adjacent sides of a quadrilateral subtend angles of 67° and 115° respectively, and the acute angle between the diagonals is 79° , find each angle of the quadrilateral.

26. If the inscribed and circumscribed circles of a triangle are concentric, prove the triangle equilateral.

Problem 1.

109. To construct a perpendicular to a given line at a given point.

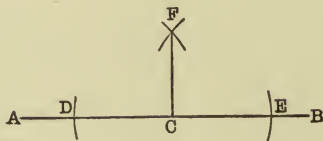


Fig. 55.

Case I: Let AB be the given line and C the given point.

Construction: With C as centre and any convenient radius, cut the line AB at D and E on opposite sides of C, with D and E successively as centres and any radius greater than DC (or CE), describe two arcs intersecting at F; a line through F and C is the required perpendicular. The first arcs described, cutting AB at D and E, are plainly for the purpose of locating two points (D and E) equally distant from C on AB, from which the two arcs (of equal radii) intersecting at F give a point equally distant from D and E. Hence we have the conditions for a perpendicular bisector for DE, which is part of AB.

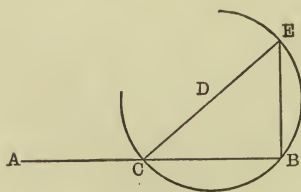


Fig. 56.

Case II: If the point is at the end of the line, say at B, take any point D, outside of AB and not immediately above B; with DB as a radius (so that the arc will pass through B), describe an arc cutting AB at C; through C and D draw a straight

line cutting AB at C; through C and D draw a straight

line, prolonging it to meet the circumference again at E; the line joining E and B will be $\perp$ to AB at B, for the angle EBC is inscribed in a semi-circle. The reasons for the procedure are evident.

Problem 2.

110. *To construct a perpendicular to a given line from a given external point.*

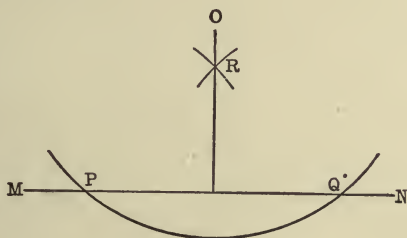


Fig. 57.

Let MN be the given line and O the given point. With O as a centre, and a radius greater than the distance from O to MN, describe an arc cutting MN in P and Q, with a larger or smaller radius (but greater than $\frac{1}{2} PQ$), and P and Q as successive centres, describe arcs intersecting at R. The line through R and O will be the required $\perp$.

It is clearly necessary to fix two points in MN equally distant from O, and then to determine another point outside MN equally distant from these two points.

111. Cor. The bisection of a line is simply a corollary to the problem.

Problem 3.

112. *To bisect a given arc.*

This is easily reduced to the bisection of a straight line by drawing the chord of the given arc, XY.

Then taking X and Y successively as centres and a radius greater than $\frac{1}{2} XY$, describe two arcs intersecting above and

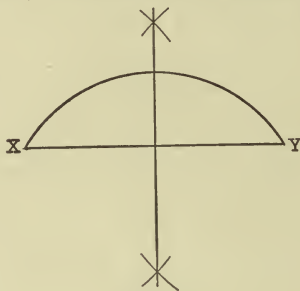


Fig. 58.

two intersecting below XY. The line joining these intersections will be a perpendicular bisector of the chord XY and hence of the arc XY.

Problem 4.

113. *To bisect a given angle.*

Let A be the given angle.

It is only necessary to bear in mind that the bisector of an arc also bisects the central angle, subtended by it, and hence the problem is practically the same as the last.

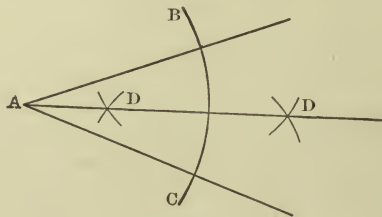


Fig. 59.

With A as a centre and any radius, describe an arc intersecting the sides of the angle in C and B. (Why is this

done?) With C and B successively as centres and a radius greater than $\frac{1}{2}$ CB, describe two arcs intersecting in D, the straight line joining D and A will bisect $\angle A$. (Why?)

Problem 5.

114. At a given point on a given straight line, to construct an angle equal to a given angle.

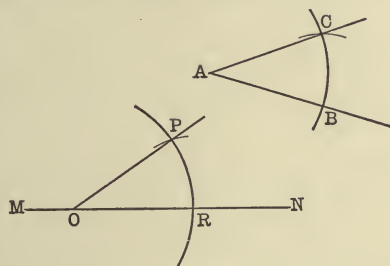


Fig. 60.

Let MN be the given line and $\angle A$ the given angle; to construct an angle at O on MN equal to $\angle A$.

Analysis: Since angles are measured by the arcs that subtend them in equal circles (Why equal circles?), it is necessary to select a radius first, that is to be used in describing the arcs, by which comparison is made, and then with the vertex of A and O successively as centres, to describe arcs which are to be used in comparing the angles.

Take any radius, and with the vertices A and O successively as centres describe arcs. The arc with centre A will cut the two sides of the angle at B and C with a radius equal to the chord of the arc BC and R, where the arc with O as a centre cuts MN as a centre cut this latter arc at P, then arc PR = arc BC. (Why?)

Hence the $\angle POR = \angle A$. (Why?)

Problem 6.

115. To draw through a given external point a line parallel to a given line.

Let O be the given point, and MN the given line; to draw through O a line parallel to MN .

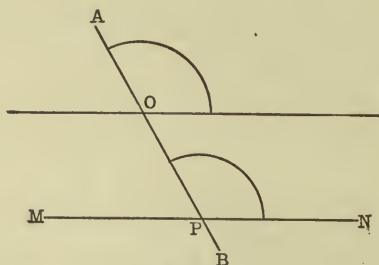


Fig. 61.

Analysis : Since two lines are $\parallel$ if they make equal exterior-interior angles with any transversal, it is suggested, immediately, to draw any transversal to MN through O and at O construct an angle equal to the angle made by this transversal with MN .

Draw through O the transversal AB , cutting MN at P . At O on AB construct (with OA as one side) an angle equal to APN . Follow Problem 5.

Problem 7.

116. To divide a straight line into any number of equal parts.

Let XY be the line; to divide it into, say, 5 equal parts.

Analysis: Bearing in mind that if any number of parallel lines cut equal parts on one transversal they cut off equal parts on every transversal, if we lay off 5 equal spaces on

any other line and, by Problem 6, draw parallels from the points of division to the given line, the problem is solved.

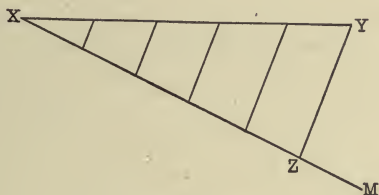


Fig. 62.

Proof: Draw any line XM from X, of indefinite length; lay off 5 equal distances (any convenient distance) on XM. Join the last point of division N on XM with the end Y, of XY, and from the other points of division on XM draw parallels to NY; they will cut off equal distances on XY. (Why?)

Problem 8.

117. To find the third angle of a triangle when two of the angles are given.

Statement: Let the angles A and B be two angles of a triangle to find the other angle.

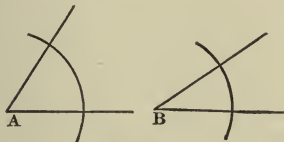


Fig. 62a.



Fig. 62b.

Analysis: The three angles of every triangle together make two right angles or one straight angle. Then what

remains of a straight angle after the two angles A and B are cut off from it must be the required angle.

Complete the construction.

Problem 9.

118. To construct a triangle having given two sides and the included angle.

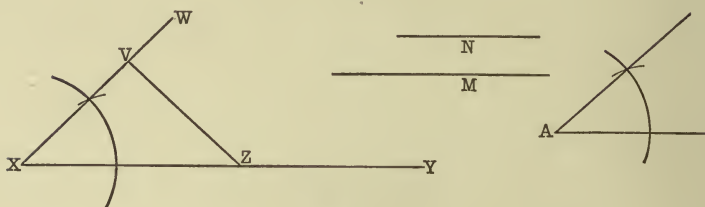


Fig. 63.

Statement: Let M and N and angle A be the given parts to construct the triangle.

Analysis: The angle will determine the directions of these two given sides, and their extremities (since their lengths are known) will determine the extremities of the third side.

Proof: Take any indefinite line XY and lay off $XZ = M$ (or N). At X construct the angle $ZXW = \angle A$, and on the side XW lay off $XV = N$ (or M, if N has been used before). Join V and Z, and XVZ will be the required triangle. (Why?)

Problem 10.

119. To construct a triangle having given two angles and the included side.

Statement: Let M and N be the given angles and O the given side. To construct the triangle.

Analysis: The angles determine the directions of the two unknown sides, and since two lines can intersect in only

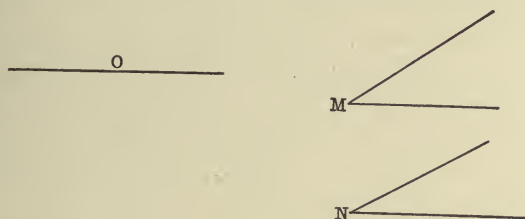


Fig. 63a.

one point, if these lines have the proper direction, they must intersect in the required point.

Complete the construction.

Problem 11.

120. *To construct a triangle having given two sides and the angle opposite one of them.*

Statement: Let M , N , and the angle O opposite N be the given parts, to construct the triangle.

Analysis: Since the given angle must be adjacent to one of the given sides, it will determine the direction of the unknown side. The other side being given in length, and starting at the other end of the given line already used, will cut this line of unknown length, but known direction, at the required point.

Proof: On the line AB of indefinite length lay off $AC = M$. At A construct on AC the angle $CAD = \angle O$.

With C as a centre and a radius equal to N , cut AD in E and F ; then either ACF or ACE will fulfill the conditions.

It is plain that if the arc described from C with radius N cuts AD , there will be two triangles that fulfill the condi-

Statement: Given the sides A and B and included angle C, to construct the parallelogram.

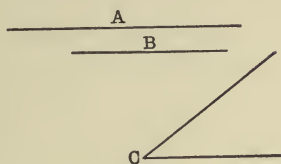


Fig. 65.

Analysis: The opposite sides of a $\square$ are parallel, hence if we know the direction of the two given sides and their lengths, we can readily draw the opposite sides (Problem 6). Since we know the angle between the given sides, we know their directions.

Complete the construction.

Problem 14.

123. To circumscribe a circle about a given triangle.

Statement: Given triangle MNO, to circumscribe a circle.

Analysis: To circumscribe a circle about MNO it must pass through the vertices M, N, O, and to secure that result the centre must be equally distant from M, N and O. The locus of points equally distant from two given points is immediately suggested.

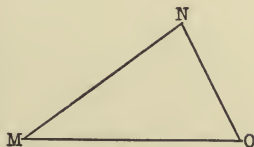


Fig. 66.

Complete the construction.

Problem 15.

124. To inscribe a circle in a given triangle.

Analysis: Here it is a case of finding a point equally distant from three given lines, which intersect, hence the locus of points equally distant from the sides of an angle is suggested.

Make the construction.

Problem 16.

125. To draw a tangent to a circle at a given point (Fig. 67).

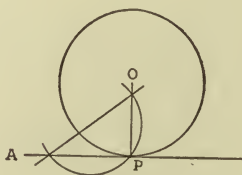


Fig. 67.

Analysis: Since a tangent is perpendicular to a radius at its extremity the suggestion is plain.

Construction: Draw radius OP to point of tangency P; a perpendicular to OP at P, its extremity (Problem I, Case II), AP is the tangent.

Problem 17.

126. To draw a tangent to a circle from an external point (Fig. 68).

Analysis: Since the point is outside the circle we do not know where the point of tangency will be, hence cannot

draw a radius to that point, and so cannot draw a perpendicular as in Problem 16. But a right angle is inscribed in a

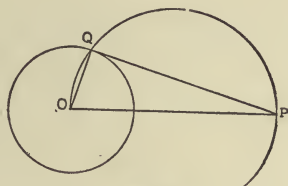


Fig. 68.

semi-circle, and this suggests drawing a semi-circle through the external point and the centre of the circle, with the line from the point to the centre as diameter.

Construction: Join the point P with the centre, O. On OP as diameter construct the semi-circumference PQO, intersecting the circle at Q. A line joining P and Q is tangent; for $\angle PQO$ being inscribed in a semi-circle is a right angle, hence PQ is perpendicular to OQ a radius.

EXERCISE III.

1. Divide a line 12'' long into seven equal parts.
2. Construct an angle of 30° ; of 45° .
3. Through a given point (outside), to draw a line making a given angle with a given line.
4. Given base and altitude of an isosceles triangle; construct it.
5. Given the base and vertical angle of an isosceles triangle; construct it.
6. Construct an equilateral triangle having one side.

7. Given the diagonals of a parallelogram and the angle between them; construct the parallelogram.

8. Construct a right triangle from its hypotenuse and one acute angle.

9. Construct a right triangle given hypotenuse and one leg.

10. Construct an equilateral triangle having its perimeter.

11. Construct an isosceles triangle having its base and the sum of its two equal sides.

12. To draw a tangent to given circle and parallel to a given line.

13. Given the altitude of an equilateral triangle; construct the triangle.

14. Through a given point in a circle draw the shortest possible chord.

15. Given an angle, an adjacent side (the base) and the altitude of a triangle; construct it. Suggestion: Draw a line $\parallel$ to the base at the altitude distance.

16. Construct a rhombus, given base and altitude.

17. Given the diagonal of a square; construct the square.

18. Given the hypotenuse of a right triangle and the length of the perpendicular upon it from the right angle; construct the triangle.

19. Describe a circle of given radius tangent to two intersecting lines.

20. In a given straight line find a point equally distant from two intersecting lines.

21. Describe a circle with a given centre and tangent to a given line.

22. Through a given point outside a given line draw a circle tangent to the line at a given point.

23. Through a given point inside a circle, to draw a chord equal to a given chord.

24. Describe a circle of given radius tangent to two given circles.

25. Describe a circle touching two intersecting lines, one of them at a given point.

26. Draw tangents common to two non-intersecting circles.

27. Construct a triangle when the mid-points of its sides are given.

28. Connect two intersecting lines by a line that will be $\parallel$ to one given line and equal to another.

29. Through a given point outside a given circle draw a secant whose internal and external segments will be equal.

30. Through a point outside a circle, to draw a secant line, so that the segment of it inside the circle shall be equal to a given line.

31. Draw a tangent to a given circle, making a given angle with a given line.

32. Show that the diameter of a circle inscribed in a right triangle equals the sum of the two legs minus the hypotenuse.

33. Construct a right triangle whose hypotenuse is three times the length of the shorter leg.

34. Construct an isosceles triangle, given base and perimeter.

35. Given, in a right triangle, an acute angle and the sum of the legs; construct it.

36. Given hypotenuse and the sum of the legs.

37. Construct a triangle, given altitude and base angles.

38. Construct a triangle, given the angles and the sum of two sides.

39. Given two circles that intersect; draw a common tangent.

40. Through one of the intersection points of two circles, to draw a straight line, so that equal parts of the line will lie inside of each circle.

41. Show by construction how much $\frac{4}{5}$ exceeds $\frac{3}{4}$.

42. Extract the square root of 5 geometrically.

APPLICATIONS OF PROPORTION.

Proposition I.

127. *If a line is drawn parallel to one side of a triangle, it divides the other two sides proportionally.*

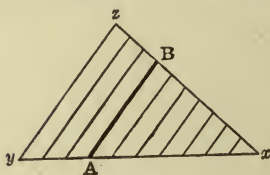


Fig. 69.

Statement: If in the triangle xyz , AB be drawn parallel to yz , then $xA : Ay :: xB : Bz$.

Analysis: Whenever a division into parts of two geometrical magnitudes is involved, there are two cases; 1st, where

the magnitudes have a common measure; 2d, where they have not.

The first case requires only a comparison of the number of times the magnitudes contain their common unit of measure; the second case requires the use of a common measure so small that the magnitudes approach commensurability.

Proof of 1st case: Find the common measure of xA and Ay , and say xA contains it m times and Ay , n times; then $xA : Ay :: m : n$.

Through the points of division draw lines parallel to yz ; then these lines divide xB into m equal parts and Bz into n equal parts.

(Why?)

Hence $xA : Ay :: m : n$ and $xB : Bz :: m : n$.

$\therefore xA : Ay :: xB : Bz$. Prove the second case.

128. Cor. I. Taking the above proportion by composition,

$$xA + Ay : xA \text{ (or } Ay) :: xB + Bz : xB \text{ (or } Bz).$$

$$\text{Or } xy : xA \text{ (or } Ay) :: xz : xB \text{ (or } Bz).$$

That is, a line $\parallel$ to one side of a triangle divides the other sides into parts which have the same ratio to those sides, respectively.

129. Cor. II. If two lines are cut by any number of parallels, their corresponding segments are proportional. That is, if the lines AB and ED are cut by the parallels, FL, GX, HY, KZ, etc., then

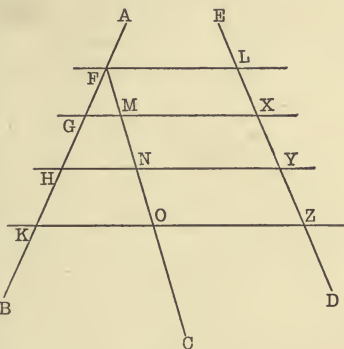


Fig. 70.

$$FG : LX :: GH : XY :: HK : YZ, \text{ etc.}$$

From F, where FL cuts AB, draw FC $\parallel$ to ED, cutting GX at M, HY at N, and KZ at O.

By Cor. I,

$$FG : FM :: FH : FN \text{ (1)}$$

$$GH : MN :: FH : FN \text{ (2)}$$

$$FH : FN :: FK : FO \text{ (3)}$$

and

$$HK : NO :: FK : FO \text{ (4)}$$

From (1) and (2)

$$FG : FM :: GH : MN.$$

From (2), (3) and (4)

$$GH : MN :: HK : NO.$$

$$\therefore FG : FM :: GH : MN :: HK : NO$$

but $FM = LX$, $MN = XY$ and $NO = YZ$. (Why?)

$$\therefore FG : LX :: GH : XY :: HK : YZ.$$

Proposition II.

130. If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.

Statement: If AB divides the sides MN and MO of the triangle MNO, so that $MN : MB :: MO : MA$, it is $\parallel$ to NO.

Analysis: If a line is drawn parallel to NO through A (say), we know that it will divide the sides proportionally, hence it is only necessary to identify AB with this line.

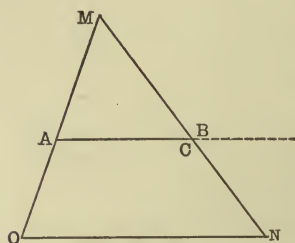


Fig. 71.

Proof: Through A draw a line $\parallel$ to NO, cutting MN at C (say). Then $MN : MC :: MO : MA$; but by hypothesis $MN : MB :: MO : MA$.

$\therefore MC = MB$; that is B and C are the same point, and since AB and AC have the points A and B in common, they are the same straight line.

Definition.

131. If a line AB be divided at C between A and B, it is said to be divided *internally* into the segments AC and CB. If C is not between A and B, but on the prolongation of AB,

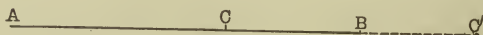


Fig. 71 a.

AB is said to be divided *externally* by C' into the segments AC' and $C'B$.

In both cases the segments are the distances from the extremities of the line AB to the point C or C' (Fig. 71 a).

If a line is divided internally and externally, so that the segments have the same ratio, the line is said to be divided harmonically.

Proposition III.

132. *The bisector of an angle of a triangle divides the opposite side into segments proportional to the including sides.*

Statement: Let AC bisect the angle A of the triangle ABD, then $BC : CD :: BA : AD$.

Analysis: We know that a line parallel to one side of a triangle divides the other sides proportionally, and since AC is the dividing line in this case, a triangle must be formed by drawing a line $\parallel$ to AC, in order to use the above principle.

Proof: From D draw DE $\parallel$ to AC intersecting AB produced at E.

By Prop. I, $BC : CD :: BA : AE$ (1).

But since DE is $\parallel$ to AC, $\angle BAC = \angle AED$ and $\angle CAD = \angle ADE$ (Why?), and $\angle BAC = \angle CAD$. $\therefore \angle ADE = \angle AED$, and hence $AE = AD$. Substituting in (1) AD for AE;

$$BC : CD :: BA : AD.$$

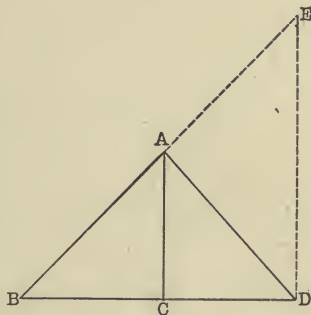


Fig. 72.

Proposition IV.

133. *The bisector of the exterior angle of a triangle divides the opposite side externally into segments proportional to the sides forming this exterior angle.*

Statement: If AB bisects the exterior angle CAF of triangle CAD, intersecting CD produced at B, then $BC : BD :: AC : AD$.

Analysis: Same as for last proposition.

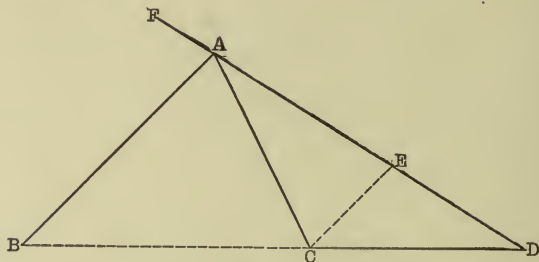


Fig. 73.

Proof: From C draw $CE \parallel$ to AB, cutting AD at E; then by Prop. I, $BC : BD :: AE : AD$ (1).

In the triangle AEC (since CE is $\parallel$ to AB), $\angle ACE = \angle BAC$ and $\angle AEC = \angle BAF$, but $\angle BAC = \angle BAF$; hence $\angle ACE = \angle AEC$, and $\therefore AE = AC$.

Substituting AC for AE in (1)

$$BC : BD :: AC : AD.$$

When is above proposition not true?

134. Cor. The bisectors of the interior and exterior angles at the same vertex divide the opposite side harmonically.

Similar Polygons.

135. Def. I. Similar polygons are those which have equal angles and their homologous sides proportional. Thus ABCDE and MNPQ are similar if $\angle A = \angle M$, $\angle B = \angle N$, $\angle C = \angle O$, etc., and $AB : MN :: AE : MQ :: ED : QP$, etc. This definition, of course, includes triangles.

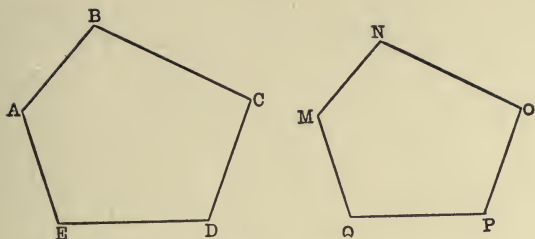


Fig. 74.

136. Def. II. Homologous parts of similar polygons are those similarly situated. In triangles, for instance, homologous sides are those opposite the equal angles.

Remark: It should be noted that polygons that differ greatly in size may be similar.

Similarity in the geometric sense applies only to form not to dimension.

Proposition V.

137. *Two mutually equiangular triangles are similar.*

Statement: If MNO and ABC are mutually equiangular,

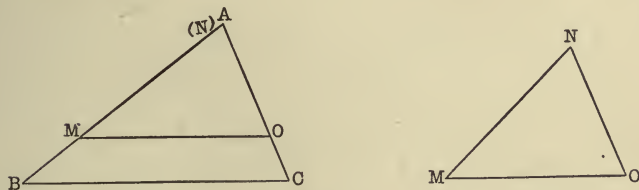


Fig. 75.

that is, if $\angle M = \angle B$, $\angle N = \angle A$, and $\angle O = \angle C$, then MNO and ABC are similar.

Analysis: Since the triangles are equiangular, it remains only to prove that their homologous sides are proportional. Since the angles are equal, the sides, including any one angle,

will take the same direction if the triangles are superposed, and then if the remaining sides can be shown to be $\parallel$, Prop. I applies.

Proof: Apply MNO to ABC so that the vertex N falls on A; then if the side MN is made to fall on AB, the side NO will fall on AC (Why?), and the side MO will take the position indicated in the figure, joining the points M and O, lying respectively on AB and AC.

Since all the angles of MNO are equal to the corresponding angles of ABC, $\angle AOM$ (corresponding to $\angle O$) $= \angle C$.

$\therefore$ MO is $\parallel$ to BC. (Why?)

Hence $AM : AB :: AO : AC$. (Why?)

But $AM = NM$ and $AO = NO$.

$\therefore NM : AB :: NO : AC$.

If the $\angle O$ is made to coincide with $\angle C$ it can be shown in the same way that $MO : BC :: NO : AC$.

138. Cor. I. Two triangles are similar if two angles of the one are equal to two angles of the other. (Why?)

139. Cor. II. Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.

Proposition VI.

140. *If two triangles have an angle of one equal to an angle of the other, and the including sides proportional, they are similar.*

Statement: If in the triangles xyz and ABC , $\angle x = \angle A$ and $xy : AB :: xz : AC$, xyz and ABC are similar.

Analysis: Since one angle in each is equal, the including sides will take the same directions, when the triangles superposed, the third side of the smaller cutting the two sides including the equal angle of the other, proportionally

by hypothesis, for the including sides in the smaller triangle will form segments of the corresponding sides of the larger.

Proof: Apply xyz to ABC , making angle x coincide with $\angle A$, and hence the side xz falls on AC and xy on AB .

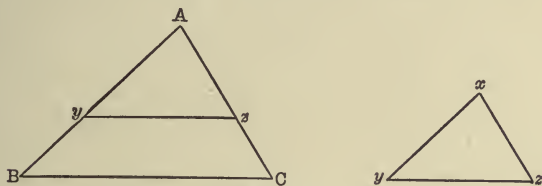


Fig. 76.

Then, since $xy : AB :: xz : AC$,
 $Ay : AB :: Az : AC$.

Hence yz is parallel to BC (Why?) ; but if yz is $\parallel$ to BC $\angle Ayz = \angle B$ and $\angle Azy = \angle C$, that is, $\angle y = \angle B$ and $\angle z = \angle C$, since Ayz is simply the triangle xyz placed on ABC .

$\therefore xyz$ and ABC are equiangular; they are similar (Prop.V).

Proposition VII.

141. *Two triangles whose sides are respectively proportional are similar.*

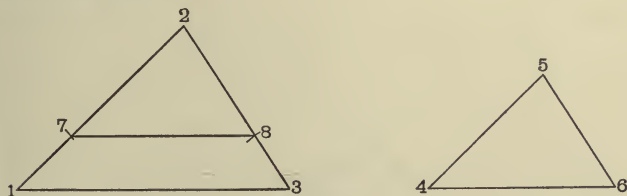


Fig. 77.

Statement: If in the triangles 123 and 456 , $12 : 45 :: 23 : 56 :: 13 : 46$, the triangles are similar.

Analysis: Since no angles are given equal, we cannot directly apply with effect one triangle to the other, but we can lay off the sides of the smaller upon those of the larger, and connecting the points thus determined, attempt to identify this triangle so constructed with the given triangle.

Proof: On 12 lay off $27 = 54$, and on 23 lay off $28 = 56$. Join 7 and 8 , forming the triangle 278 .

The triangle 278 is similar to 123 . (Why?)

Also in the triangles 278 and 123 (since they are similar)
 $21 : 27 :: 13 : 78$. (a)

But by hypothesis, $21 : 45 :: 13 : 46$. (b)

$27 = 45$ by construction, hence, by comparing (a) and (b)
 $78 = 46$.

$\therefore$ triangle 278 and triangle 456 (having 3 sides equal) are equal triangles. Hence, since 278 is similar to 123 , 456 is similar to 123 .

Proposition VIII.

142. *Two triangles having their sides respectively parallel or perpendicular are similar.*

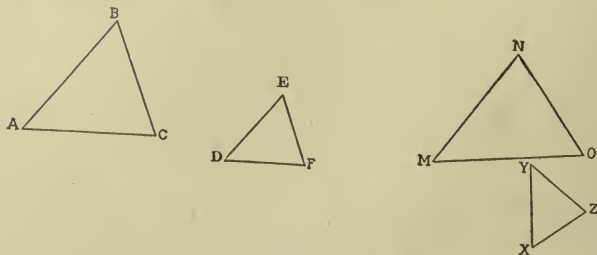


Fig. 78.

Statement: If in the triangles ABC and DEF , $AB \parallel$ to ED , $AC \parallel$ to DF , and $BC \parallel$ to EF , the triangles are similar.

Also, in the triangles MNO and XYZ , if XY is perpendi-

cular to MO, YZ perpendicular to MN, and XZ perpendicular to NO, MNO and XYZ are similar.

Analysis: Since angles whose sides are either $\parallel$ or perpendicular are equal (provided the sides extend in the same direction as these must do) (Why?), these triangles will be equiangular and hence similar.

Proposition IX.

143. *Two homologous altitudes of two similar triangles have the same ratio as any two homologous sides.*

Statement: If ABC and MNP are similar, and BD and NO are the altitudes, then $BD : NO :: AB : MN :: AC : MP :: BC : NP$.

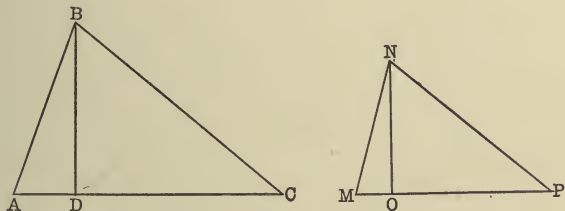


Fig. 79.

Analysis: The altitudes form two right triangles which are readily proved similar.

Proof: ABD and MNO are similar. (Why?)

$\therefore AB : MN :: BD : NO$ (1). Also BDC and NOP are similar.

$\therefore BC : NP :: BD : NO$ (2).

Also $AB : MN :: AC : MP$ (3).

Then $BD : NO :: AB : MN :: AC : MP :: BC : NP$,
[combining (1), (2) and (3)].

144. Cor. Any homologous lines in two similar triangles are in proportion to the sides.

Proposition X.

If two parallels are cut by three or more transversals that pass through the same point, the corresponding segments are proportional.

Statement: Let A be the fixed point (whether between the parallels or on the same side of both), and BF and GM be the parallels cutting the transversals, at B, C, D, E and F, and G, H, K, L and M respectively.

Analysis: The similar triangles are evident.

Proof: Since ABC and AGH are similar (Why?), $BC : GH :: AC : AH$. Also since ACD and AHK are similar, $AC : AH :: CD : HK$. $\therefore BC : GH :: CD : HK$.

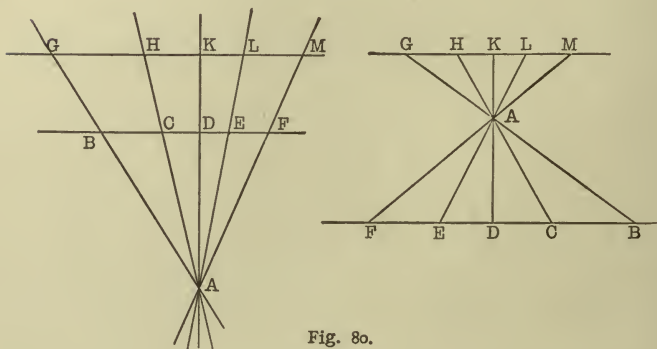


Fig. 80.

In the same way may be proved that $CD : HK :: DE : KL$, etc.

Hence $BC : GH :: CD : HK :: DE : KL :: EF : LM$.

Proposition XI.

145. *If three or more non-parallel transversals intercept proportional segments upon two parallels, they pass through a common point.*

Statement: If BE, CF and DG cut MN and RS at B, C, D and E, F, G respectively so that $BC : EF :: CD : FG$, then BE, CF, DG produced, will meet in a point, say A.

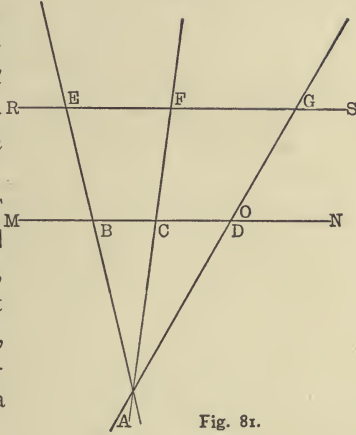


Fig. 81.

Analysis: Two non-parallel lines will always meet if produced, and if a third line be drawn through this point and one of the intersection points of a third given transversal, this third given transversal may be identified with the arbitrarily drawn third line, and hence it will pass through A.

Proof: Produce BE and CF to meet in, say, A. Through A and G draw arbitrarily a line, which meets MN, say, at O. Then by Prop. IX,

$$BC : EF :: CO : FG,$$

but by hypothesis, $BC : EF :: CD : FG$.

$\therefore CO = CD$, that is, the points O and D are the same; hence the arbitrary line AOG is the same as the given line DG.

$\therefore$ DG will pass through A.

Proposition XII.

146. *The perimeter of two similar polygons have the same ratio as any two homologous sides.*

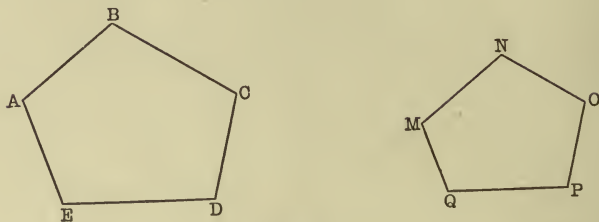


Fig. 82.

Statement: If $ABCDE$ and $MNOPQ$ are similar, then $AB + BC + CD + DE + AE : MN + NO + OP$, etc., $:: AB : MN :: AE : MQ$, etc.

Analysis: The sides form a series of equal ratios; hence by the theory of proportion, the antecedents and consequents may be combined.

Proof: Since the polygons are similar by definition,

$$AB : MN :: AE : MQ :: ED : QP, \text{ etc.}$$

$\therefore (AB + AE + ED + \text{etc.}) : (MN + MQ + QP + \text{etc.}) :: AB : MN :: AE : MQ$, etc., by theory of proportion.

Proposition XIII.

147. *If in a right triangle a perpendicular is drawn from the vertex of the right angle to the hypotenuse:*

The triangles thus formed are similar to each other and to the given triangle.

The perpendicular is the mean proportional between the segments of the hypotenuse.

Each leg of the right triangle is the mean proportional between the hypotenuse and the adjacent segment.

Statement : Let AC be a $\perp$ from the right angle A upon the hypotenuse BD, then triangles BAC and ACD are similar to each other and to the triangle BAD; also $BC : AC :: AC : CD$. $BD : AB :: AB : BC$ and $BD : AD :: AD : CD$.

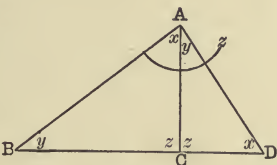


Fig. 83.

Analysis: To prove two triangles similar, it is usually easiest to show that they have equal angles. After they are proved similar, their sides are known to be proportional.

Proof: In triangle ABC and triangle ACD, $\angle BAC = \angle ADC$ (give two reasons). $\therefore$ since they are right triangles, triangle ABC is similar to triangle ACD. Triangle ABC and triangle ABD have $\angle B$ common, $\therefore$ they are similar, and triangle ACD and ABD have $\angle D$ common, $\therefore$ they are similar.

In the similar triangles ABC and ACD, $BC : AC :: AC : CD$. Also in the similar triangles ABC and ABD (see note) $BD : AB :: AB : BC$. In the similar triangles ACD and ABD, $BD : AD :: AD : CD$.

NOTE. — To make the relations clear this proportion may be written thus: BC (opposite x in ABC) : AC (opposite x in ACD) :: AC (opposite y in ABC) : CD (opposite y in ACD). Likewise, BD (opposite z in BAD) : AB (opposite z in ABC) :: AB (opposite x in ABD) : BC (opposite x in ABC).

It will be observed that homologous sides of similar triangles are those opposite equal angles, hence it facilitates the selection of homologous sides to mark the equal angles with the same letter, as x , y or z .

148. Cor. I. The perpendicular from any point in the circumference of a circle is a mean proportional between the segments of the diameter.

149. Cor. II. The square of the hypotenuse is equal to the sum of the squares of the other two sides.

For $BD : AB :: AB : BC$ or $BD \times BC = \overline{AB}^2$

and $BD : AD :: AD : CD$ or $BD \times CD = \overline{AD}^2$

add $BD (BC + CD) = \overline{AB}^2 + \overline{AD}^2$.

But $BC + CD = BD$.

$$\therefore \overline{BD}^2 = \overline{AB}^2 + \overline{AD}^2.$$

150. Definition. The projection of a point on a line is the foot of the perpendicular from the point to the line. The projection of one line on another is the segment of the second line included between the feet of the perpendiculars upon it from the extremities of the first line.

If the lines intersect, the projection of either line upon the other is the distance from the point of intersection to the foot of the perpendicular, from the end of the projected line upon the other.

Proposition XIV.

151. *In any triangle the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides diminished by twice the product of one of those sides by the projection of the other upon that side.*

Statement: In the triangle ABD, when CD is the projection of BD on AD, then

$$\overline{AB}^2 = \overline{AD}^2 + \overline{BD}^2 - 2 AD \times CD.$$

Analysis: In cases of this sort it is the most direct process to find the direct value of the line involved (here, AB^2) by the most evident process and then to reduce this value to the required form.

Proof: $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$. . . (a) (By Prop. XIII, Cor. II.)

But $AC = AD - CD$. (This expression is taken to bring in the whole line AD and the other segment CD , both of which are given in the required expression.)

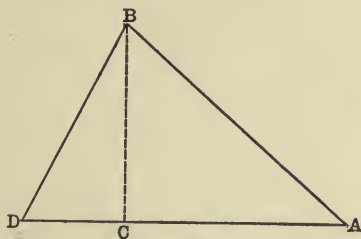


Fig. 84.

$$\therefore \overline{AC}^2 = \overline{AD}^2 + \overline{CD}^2 - 2 AD \times CD.$$

Substituting this value of $\overline{AC}^2$ in (a), $\overline{AB}^2 = \overline{AD}^2 + \overline{CD}^2 + \overline{BC}^2 - 2 AD \times CD$. But $\overline{CD}^2 + \overline{BC}^2 = \overline{BD}^2$ (Why?).
 $\therefore \overline{AB}^2 = \overline{AD}^2 + \overline{BD}^2 - 2AD \times CD$.

Cor. If the triangle contains an obtuse angle, then the square of the side opposite the obtuse angle equals the sum of the squares of the other two sides plus the product of one of those sides by the projection of the other upon it.

The proof is exactly similar, except that here one segment of the side upon which projection is made is equal to the whole side plus other segment.

Proposition XV.

152. *The sum of the squares of two sides of a triangle is equal to twice the square of half the third side plus twice the square of the median upon that side.*

The difference of the squares of two sides of a triangle is equal to twice the product of the third side by the projection of the median upon it.

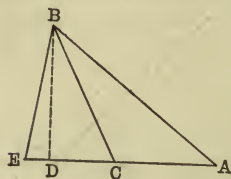


Fig. 85.

Statement: In the triangle ABE, let BC represent the median and BD be a $\perp$ from B upon AE, then CD is the projection of BC upon AE.

Then

$$\overline{AB}^2 + \overline{BE}^2 = 2 \overline{BC}^2 + 2 \overline{AC}^2 \text{ and } \overline{AB}^2 - \overline{BE}^2 = 2 AE \times CD.$$

Analysis: Same as last proposition.

Proof:

$$\overline{BE}^2 = \overline{BC}^2 + \overline{CE}^2 - 2 CE \times CD \text{ (Prop XIV).}$$

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + 2 AC \times CD \text{ (Prop. XIV, Cor.)}$$

Add, remembering that CE = AC,

$$\overline{BE}^2 + \overline{AB}^2 = 2 \overline{BC}^2 + 2 \overline{AC}^2.$$

Subtract ;

$$\overline{AB}^2 - \overline{BE}^2 = 2 CE \times CD + 2 AC \times CD = CD \times 4 AC = CD \times 2 AE.$$

The median, which is a line drawn from one vertex in a triangle to the middle of the opposite side, should be carefully distinguished from the bisector of an angle, which does not usually meet the middle of the side opposite to the angle.

Proposition XVI.

153. *If two chords intersect in a circle, the product of the segments of one is equal to the product of the segments of the other.*

Statement: Let the chords MN and PQ in the circle C, intersect at O, then to prove $MO \times ON = OP \times OQ$.

Analysis: The required equation, $MO \times ON = OP \times OQ$, suggests the product of means and extremes, and hence a proportion, and a proportion suggests similar triangles.

Then it is necessary to form similar triangles, which contain the lines MO, ON, OP, and OQ as sides. The auxiliary lines required are evident.

Proof: Connect M with P, and N with Q, then in the two triangles MOP and NOQ:

$$\begin{aligned}\angle MOP &= \angle NOQ. & (\text{Why?}) \\ \angle PMO &= \angle NQO. & (\text{Why?}) \\ \angle MPO &= \angle ONQ. & (\text{Why?})\end{aligned}$$

$\therefore$ MOP is similar to NOQ. Mark equal angles as x , y and z in both.

Then PO (opposite x in MOP) : ON (opposite x in NOQ)
 $::$ MO (opposite y in MOP) : OQ (opposite y in NOQ).

$$PO : ON :: MO : OQ.$$

$\therefore PO \times OQ = ON \times MO$ (equating products of extremes and means).

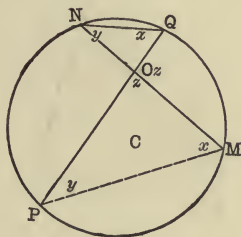


Fig. 86.

Proposition XVII.

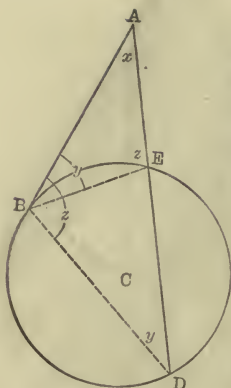


Fig. 87.

154. If from a point without a circle, a secant and a tangent are drawn, the tangent is the mean proportional between the whole secant and the part of it without the circle (its external segment).

Statement: Let A be the point, AD the secant, and AB the tangent to circle C.

Then

$AD : AB :: AB : AE$. (AE being part of secant outside circle.)

Analysis: Same as for last proposition.

Proof: Join B with E, and B with D.

Then in the $\triangle$'s ABE and ABD.

$$\angle A = \angle A.$$

$$\angle ABE = \angle ADB. \quad (\text{Why?})$$

$$\angle AEB = \angle ABD \quad (\text{Why?}), \text{ marking } x, y \text{ and } z.$$

Then AD (opp. z in ABD) : AB (opp. z in ABE) : : AB (opp. y in ABD) : AE (opp. y in ABE) or $AD \times AE = \overline{AB}^2$.

Remark: The length of a secant is understood to mean the distance from the external point to the second point of intersection with the circle.

EXERCISE IV.

1. Show that in a given circle the product of the parts into which a fixed point divides any chord drawn through it, is constant.

2. If the sides of a quadrilateral are bisected prove that the lines joining these middle points in order form a parallelogram.

Suggestion: Draw the diagonals.

3. With a pole ten feet long, show how to find the height of a steeple on a clear day.

4. With sufficient level space on the bank of a stream show how to find its width, by using a tape line only.

5. Prove that the lines joining the feet of $\perp$'s from the vertices of a parallelogram upon the opposite diagonals, themselves form a parallelogram.

6. Show that the medians of a triangle meet in a point.

7. The common external tangent of two circles, touching externally, is a mean between their diameters.

8. If a quadrilateral is inscribed in a circle, the product of its diagonals equals the sum of the products of opposite sides.

9. If three lines be drawn through the point of tangency of two circles, the triangle formed by joining their other points of intersection in each circle is similar.

10. In the parallelogram mnp , pr is drawn, meeting the side mn , produced at r ; the diagonal om at s and on at t .

Prove:

$$ps^2 = st \times sr.$$

11. Tangents drawn to two intersecting circles forming a point on their common chord produced, are equal.

12. Two circles intersect. In each circle a chord is drawn from one of the points of intersection, tangent to the other circle. Show that the common chord is a mean proportional between the lines joining the other point of intersection with the ends of the tangent chords.

13. Determine a point in the circumference of a circle from which if chords be drawn to two fixed points, these chords will have the ratio 2 : 3.

14. In a given circle a line is drawn perpendicular to the diameter produced. If from the remoter end of the diameter, a secant be drawn to any point of the perpendicular, the product of the whole secant by its internal segment is constant.

15. To pass a circle through two given points tangent to a given line.

16. If there be two luminous points, whose powers of illuminating a point equally distant from each are as 3 : 4, find the locus of points in their plane equally illuminated.

17. If the middle points of the two pairs of adjacent sides of a parallelogram are joined by straight lines, show that the diagonal is trisected.

18. Through a given point within a given circle draw a chord which shall be divided in the ratio 1 : 2 by the point.

19. Describe a circle whose circumference shall pass through a given point and cut two given circles orthogonally, that is, at right angles.

20. If through the point of tangency of two circles three lines are drawn, prove that the triangles formed by joining the three intersection points in each circle are similar.

21. The sides of a triangle are 8 ft., 15 ft., and 17 ft., the shortest side of a similar triangle is 5 ft., find the remaining sides. What kind of a triangle is it?

22. Find the product of the segments of a chord drawn through a point 5 in. from the centre of a circle whose radius is 9 in.

23. The diameter which bisects a chord 8 ft. long, is $2\frac{5}{3}$ ft. in length. What is the distance from the end of the chord to the end of the diameter?

24. The distance of the moon from the earth is approximately 239,000 miles. If a disk 1 in. in diameter placed 10 ft. from the eye just covers the moon, what is the moon's diameter.

25. If the diameter of the earth is 7,918 miles, what is the distance from the eye of an observer, raised 1 mile above the surface, to the most remote visible point on the earth's surface?

26. The horizontal distance from A to B is 12.84 ft., and the heights of A and B above a given level are 47.32 ft. and 55.91 ft. respectively. Find the height of points whose horizontal distances from A are 1 ft. and 2 ft. respectively.

27. Divide a hexagon into 4 equal and similar parts.

28. Devise a method of showing to the eye that the sum of the squares upon the base and vertical leg of a right triangle equals the square on the hypotenuse.

29. A strip of carpet 1 yard wide is laid diagonally across a room 18 ft. square. What is the length of the strip, if it just touches the walls at its corners?

30. The sides of a triangle are 24 in., 25 in., and 18 in. Find the segments into which the bisector of the angle, between the 24 in. and 25 in. sides, divides the 18 in. side.

31. Find the lengths of the medians in Ex. 30.

32. The middle point of a chord is distant 5 in. from the middle of its subtended arc. Find the diameter of the circle if the chord is 20 in. long.

33. If another chord in the same circle is distant 9 in. from the middle of its subtended arc, what is the length of the chord?

34. The length of one diagonal of a trapezoid is 30 in., and the segments into which it divides the other diagonal are 12 in. and 6 in. What are the lengths of the segments of the first diagonal?

35. If two of the adjacent sides of a parallelogram are 5 in. and 12 in., and the diagonal drawn from their point of intersection is 17 in., find the other diagonal.

36. The diameters of two circles are 12 in. and 28 in., respectively, and the distance between their centres is 29 in. What is the length of their common internal tangent.

37. Show that the square of the common tangent to two circles (unequal) touching each other externally equals four times the product of their radii.

38. The diameter of a circle is 18 in. One segment of a chord drawn through a point 7 in. from the centre is 4 in. Find the other segment.

39. The diameter of a circle is 24 in.; if tangents are drawn from a point 20 in. from the centre, what is their length and what is the length of the chord of contact?

40. A circle has a radius of 7 in. From the end of a tangent 24 in. long a secant is drawn through the centre. What is the length of the secant?

41. A chord of a circle is 20 in. long and the sagitta of its arc is 2 in. What is the radius.

42. The span of a circular arch is 120 ft., and the radius of the circle of which it is an arc is 725 ft. Find the height of the middle of the arch.

AREAS.

155. *Definition.* The area of a surface is the number of times it contains the unit of area.

The square whose side is a linear unit in length is accepted as the unit of area.

That is, if the area of a surface is to be expressed in square inches, a square whose side is one inch is the unit. If the area of the surface is to be expressed in square miles,

a square, whose side is one mile is taken as the unit. Evidently a square of any size desired may be taken as a unit, and the area expressed in the square of that unit.

Proposition I.

156. *The area of a rectangle equals the product of its base by its altitude.*

That is, the area of a rectangle in square units equals the product of the number of linear units, in its base and altitude.

Statement: If the base of the rectangle ABDC is CD and its altitude BD, then area of ABCD = $BD \times CD$.

Analysis: There are plainly two cases; when the base and altitude contain a common unit, that is, are commensurable; and when they do not, or are incommensurable. In the first case it is only neces-



Fig. 88.

sary to divide the base and altitude into as many parts as they will contain their common unit, and by drawing through these points of division, lines $\parallel$ to the base and altitude respectively, to divide the rectangle into squares.

In the 2d case, a unit is chosen so small that the base and altitude approach commensurability as near as desired.

Proof: Apply the common unit to BD and CD. Say it is contained in BD, m times, and in CD, n times. Through these points of division draw lines $\parallel$ to BD and CD respectively. The rectangle will then be divided into n

tiers, containing m squares each, that is mn square units.
 $\therefore$ area of $ABCD = mn = BD \times CD$.

Case II. Say the unit of length chosen is contained in BD , m times, and in CD , n times plus a remainder ED , less than the unit. Through E draw $EF \parallel$ to BD , then $ACEF = EF \times CE$ or $BD \times CE$ (since $EF = DB$).

Since the remainder is always less than the divisor, if we take a unit $\frac{1}{4}$ as large as before, it will be contained in BD and CE an even number of times and will doubtless reduce the remainder ED . If a unit of indefinite smallness be used, the remainder ED will be still smaller, and it is evident that this remainder may be made as small as desired, by taking a unit small enough. But always, $ACEF = BD \times CE$, hence, since E can be made to approach D as near as desired, eventually $ACDB = BD \times CD$.

157. Cor. I. Two rectangles having one dimension equal, are to each other as the other dimension.

158. Cor. II. Two rectangles are to each other as the product of their two dimensions.

Proposition II.

159. *The area of a parallelogram equals the product of its base and altitude.*

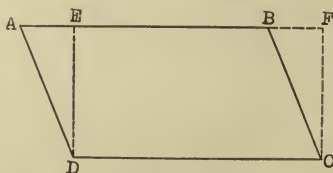


Fig. 89.

Statement: If DC is the base and DE is the altitude of the parallelogram $ABCD$, then area of $ABCD = DC \times DE$.

Analysis: If a rectangle be constructed on DC as base and with the altitude DE , by producing AB and dropping a perpendicular CF upon it from C , the area of the parallelogram may be compared with this rectangle having the same base and altitude.

Proof: Construct the rectangle EDCF as indicated.

The area of EDCF = $DC \times ED$.

In the right triangles AED and BFC.

$$ED = FC. \quad (\text{Why?})$$

$$AD = BC. \quad (\text{Why?})$$

$$\therefore \text{triangle AED} = \text{triangle BFC}. \quad (\text{Why?})$$

But the quadrilateral ADCF – triangle BFC = parallelogram ABCD;

and the quadrilateral ADCF – triangle AED = rectangle EDCF.

$$\therefore \square ABCD = \text{rectangle EDCF}. \quad (\text{Why?})$$

But $\square ABCD$ has same base and altitude as rectangle EDCF, and $EDCF = DC \times ED$. $\therefore ABCD = DC \times ED$.

State Corollaries like those for Prop. I.

Proposition III.

160. *The area of a triangle equals one-half the product of its base and altitude.*

Statement: If ON is the base and MR the altitude of the triangle MNO, then area of MNO = $\frac{1}{2}$ (ON $\times$ MR).

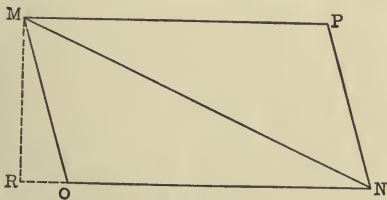


Fig. 90.

Analysis: Recalling the relation between a parallelogram and the two triangles into which it is divided by a diagonal, the procedure is evident.

Proof: On ON as base and with MR as altitude, construct a parallelogram OMPN by drawing $MP \parallel$ and equal to ON, and joining P and N. MN will evidently be the diagonal of this parallelogram.

Triangle $OMN = \frac{1}{2}$ parallelogram $OMPN$, but $\square$
 $OMPN = ON \times MR$ (product of base and altitude).

$$\therefore \text{triangle } OMN = \frac{1}{2} (ON \times MR).$$

161. Cor. I. By the associative law of multiplication,
 $\frac{1}{2} (ON \times MR) = \frac{1}{2} ON \times MR = ON \times \frac{1}{2} MR$; that is,
 the area of a triangle equals the product of one-half the base
 by the altitude, or the product of one-half the altitude by
 the base.

State corollaries similar to those under Prop. I.

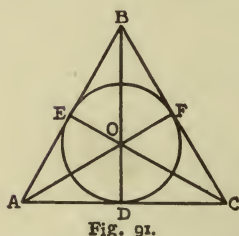


Fig. 91.

162. Cor. II. The area of tri-
 angle equals one-half the perimeter
 by the radius of the inscribed circle.

Call p the perimeter of ABC
 and r the radius, of the inscribed
 circle O . Draw radii, OD , OE
 and OF , to the points of contact of
 this circle, with sides, AC , AB and

BC respectively. Also join O with A , B and C .

Then OD is the altitude of the triangle AOC }
 OE is the altitude of the triangle AOB } (Why?)
 and OF is the altitude of the triangle BOC }

Hence area $AOC = \frac{1}{2} (AC \times OD) = \frac{1}{2} AC \times OD$;

area $AOB = \frac{1}{2} (AB \times OE),$

and area $BOC = \frac{1}{2} (BC \times OF)$

adding; area $ABC = \frac{1}{2} (AC + AB + BC) \times OD$ (since
 $OD = OE = OF = r$)

or area $ABC = \frac{1}{2} p \times r.$

Proposition IV.

163. *The area of a trapezoid equals the product of its
 altitude by one-half the sum of its parallel sides.*

Statement: If AB and CD are the two parallel sides of

the trapezoid ABCD, and BE its altitude, then area of $ABDC = \frac{1}{2} (AB + CD) \times BE$.

Analysis: A diagonal will clearly divide the trapezoid into two triangles, both having the altitudes of the trapezoid, and the parallel sides for bases, respectively.

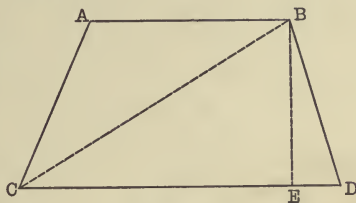


Fig. 92.

Proof: Draw the diagonal CB. The triangle ABC has base AB and altitude equal to BE, and the triangle CBD has base CD and altitude BE.

$$\text{Area triangle ABC} = \frac{1}{2} AB \times BE.$$

$$\text{Area triangle CBD} = \frac{1}{2} CD \times BE.$$

$$\text{Add; trapezoid ABDC} = \frac{1}{2} (AB + CD) \times BE.$$

164. Cor. The area of a trapezoid equals the product of the altitude by a line joining the middle points of the two non-parallel sides. Prove it.

Proposition V.

165. *The areas of two triangles having an equal angle are to each other as the product of the sides including the equal angles.*

Statement: In the triangles HKL and MNO let $\angle H = \angle M$, then

$$\frac{\text{area HKL}}{\text{area MNO}} = \frac{HK \times HL}{MN \times MO}.$$

Analysis: Since these triangles have an equal angle, it is immediately suggested to superpose them for comparison, since we know that the two sides, including the equal angles in each, will take the same position. The next

suggestion is just as clear: to draw a connecting line between the non-coincident sides. These two steps are clearly suggested in all similar cases.

Proof: Place vertex M on H, making MN fall on HK, then MO will fall on HL (Why?), and triangle MNO will

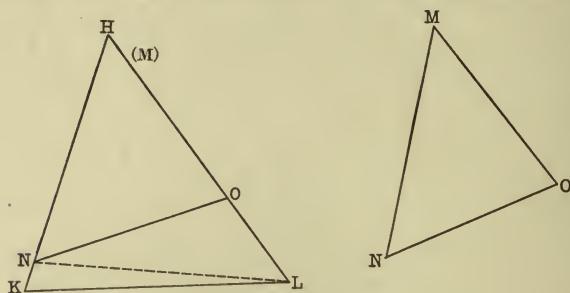


Fig. 93.

take the position HNO. Connect N and L (or O and K) by straight line. Then HNL is a triangle intermediate between HNO and HKL.

Triangles HNO and HNL have the same altitude (a perpendicular from N to HL, taking HO and HL as bases).

$$\therefore \frac{\text{HNL}}{\text{HNO}} = \frac{\text{HL}}{\text{HO}} \quad (\text{a}) \quad (\text{Why?})$$

Likewise triangles HNL and HKL have same altitude (a perpendicular from L to HK, taking HN and HK as bases).

$$\therefore \frac{\text{HKL}}{\text{HNL}} = \frac{\text{HK}}{\text{HN}} \quad (\text{b})$$

Multiply (a) by (b).

$$\frac{\text{HNL}}{\text{HNO}} \times \frac{\text{HKL}}{\text{HNL}} = \frac{\text{HL}}{\text{HO}} \times \frac{\text{HK}}{\text{HN}}$$

that is, $\frac{\text{HKL}}{\text{HNO}} = \frac{\text{HL} \times \text{HK}}{\text{MO} \times \text{MN}}$ (since $\text{HO} = \text{MO}$ and $\text{HN} = \text{MN}$ by superposition).

166. Cor. I. Similar triangles are to each other as the squares of any two homologous sides.

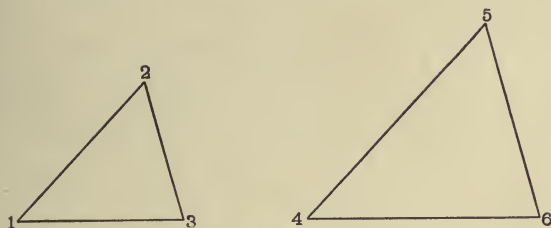


Fig. 94.

Let triangle $123 \approx$ triangle 456 [$\approx$ = similar to.]

Then
$$\frac{123}{456} = \frac{12^2}{45^2} = \frac{13^2}{46^2} = \frac{23^2}{56^2}.$$

For $\frac{123}{456} = \frac{12 \times 13}{45 \times 46} = \frac{12}{45} \times \frac{13}{46}$ (since being similar their angles are equal),

but
$$\frac{12}{45} = \frac{13}{46}. \quad (\text{Why?})$$

$$\therefore \frac{123}{456} = \frac{12}{45} \times \frac{13}{46} = \frac{12^2}{45^2} \left(\text{or} = \frac{13}{46} \times \frac{13}{46} = \frac{13^2}{46^2} \right).$$

Likewise,
$$\frac{123}{456} = \frac{13 \times 32}{46 \times 65} = \frac{13}{46} \times \frac{32}{65} = \frac{32^2}{65^2} \text{ etc.}$$

167. Cor. II. Similar polygons are to each other as the squares of any two homologous sides. Prove it.

Suggestion: Divide the polygon up into triangles by diagonals from any vertex. These triangles will be similar. Apply principle of continued proportion.

REGULAR POLYGONS AND CIRCLES.

168. Definition: A regular polygon is a polygon which is equilateral and equiangular.

Evidently an equilateral polygon inscribed in a circle is a regular polygon; so that every polygon, formed by dividing a circumference into equal parts, and drawing the chords of these equal arcs, is regular.

Proposition I.

169. Regular polygons of the same number of sides are similar.

Analysis: Since the sum of the interior angles of a polygon depends entirely upon the number of sides (What is the sum of the interior angles of a polygon of n sides?), these polygons will have the same total of interior angles, and since they are regular the individual angles will be equal. Hence they have equal angles. The polygons being regular are also equilateral; hence all sides have the same ratio. Draw figure and make proof definite.

170. Cor. Since regular polygons of the same number of sides are similar, their perimeters have the same ratio as any two homologous lines; hence the perimeters of regular polygons of the same number of sides, are to each other as the radii of the inscribed circles, or as the radii of the circumscribed circles.

Proposition II.

171. Two circumferences have the same ratio as their radii.

Statement: Let the circles A and B have radii R and R' respectively, then circumference of A : circumference of B :: $R : R'$.

Analysis: The proportion suggests the relation between the perimeters of regular polygons of the same number of sides, and regular polygons of the same number of sides may be inscribed in both circles. The number of sides may be infinite without altering the relation.

Proof: Inscribe the regular polygon of perimeter P in A and the regular polygon of perimeter P' of same number of sides in B , then $P : P' :: R : R'$, however many sides P and P' may contain.

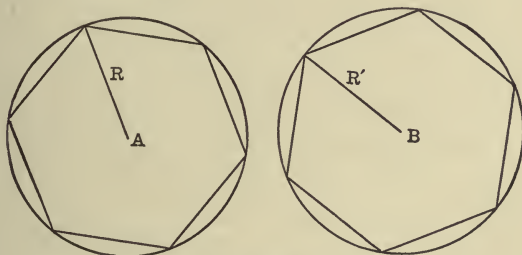


Fig. 95.

If the number of sides be infinitely great, the perimeters of the polygons coincide with the circumferences of the circles, hence eventually $C : C' :: R : R'$ (where C and C' represent the circumferences of A and B respectively).

Cor: The ratio of the circumference of every circle to its diameter is the same.

For let C and C' be the circumference of any two circles whose radii are R and R' ,

then, $C : C' :: R : R'$,

or $C : C' :: 2R : 2R'$.

By alternation, $C : 2R :: C' : 2R'$,

that is, $\frac{C}{2R} = \frac{C'}{2R'}$ or $\frac{C}{D} = \frac{C'}{D'}$.

It is customary to call this ratio π .

Hence, $\frac{C}{2R} = \pi$; $C = 2\pi R$.

Proposition III.

172. *The area of a regular polygon is equal to half the product of its perimeter and apothem (apothem meaning the perpendicular from the centre upon any side).*

Statement: Call P the perimeter of a regular polygon and h its apothem, then its area $A = \frac{1}{2} h \times P$.

Analysis: Any regular polygon can be divided into as many equal isosceles triangles as it has sides by drawing radii from the centre to the vertices. The apothem is the common altitude of these triangles, and the sides of the polygon are the bases of the triangles.

Complete the proof.

Proposition IV.

173. *The area of a circle is equal to half the product of its radius by its circumference.*

Statement: Let A be the area, R the radius and C the circumference of the circle, then $A = \frac{1}{2} R \times C$.

Analysis: Inscribe a regular polygon and apply Prop. III.

Proof: Inscribe a regular polygon with perimeter P and apothem h . Then the area of the polygon $A' = \frac{1}{2} h \times P$, no matter how many sides it may have.

If the number of sides is sufficiently increased, the perimeter of the polygon approaches the circumference, that is, P approaches C and h approaches R ; hence ultimately

$$A = \frac{1}{2} R \times C = \frac{1}{2} R \times 2\pi R = \pi R^2.$$

Proposition V.

174. *Given the side of a regular inscribed polygon and the radius of the circle to find the side of the regular inscribed polygon of double the number of sides.*

Statement: Let AB be the given side of an inscribed regular polygon of n sides, AC the side of a regular inscribed polygon of $2n$ sides, R the radius and E the centre. Call AB, x . To find AC in terms of R and x draw diameter $CF \perp$ to AB , intersecting AB at D . Then D is the middle of AB (Why?), and hence $AD = \frac{1}{2}x$; connect A and F .

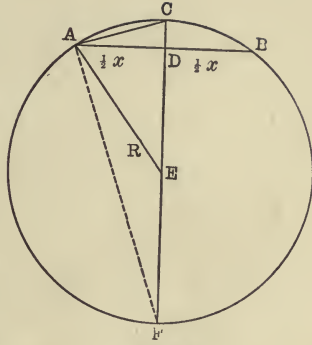


Fig. 96.

Solution:

$$DE^2 = AE^2 - (\tfrac{1}{2}x)^2 \text{ or } DE = \sqrt{R^2 - \frac{x^2}{4}}.$$

$$CD = R - DE = R - \sqrt{R^2 - \frac{x^2}{4}};$$

also, $CF : AC :: AC : CD$ or $\overline{AC}^2 = CF \cdot CD$. (Why?) Observe that CAF is a right triangle and AD a perpendicular on the hypotenuse).

$$\therefore \overline{AC}^2 = 2R \left(R - \sqrt{R^2 - \frac{x^2}{4}} \right).$$

$$\begin{aligned} \therefore AC &= \sqrt{2R \left(R - \sqrt{R^2 - \frac{x^2}{4}} \right)} \\ &= \sqrt{R \left(2R - \sqrt{4R^2 - x^2} \right)}. \end{aligned}$$

If $R = .1$ for simplicity, $AC = \sqrt{2 - \sqrt{4 - x^2}}$.

175. *Cor.* To compute π . Take $R = 1$ then $C = 2\pi$
or $\pi = \frac{1}{2} C$.

In the formula let AB be a side of a regular hexagon,
hence, $AB = R = 1$ Say $AB = S_6$, $AC = S_{12}$, etc.

$$\begin{aligned} \text{Then } S_{12} &= \sqrt{2 - \sqrt{4 - 1}} = .51763809 +; \\ &\text{hence } P_{12} = 6.21165708 + \end{aligned}$$

$$\begin{aligned} S_{24} &= \sqrt{2 - \sqrt{4 - .51763809}} = .26105238 +; \\ &\text{hence } P_{24} = 6.26525722 + \end{aligned}$$

$$\begin{aligned} S_{48} &= \sqrt{2 - \sqrt{4 - .26105238}} = .13080626 +; \\ &\text{hence } P_{48} = 6.27870041 + \end{aligned}$$

$$\begin{aligned} S_{96} &= \sqrt{2 - \sqrt{4 - .13080626}} = .06543817 +; \\ &\text{hence } P_{96} = 6.28206396 + \end{aligned}$$

$$\begin{aligned} S_{192} &= \sqrt{2 - \sqrt{4 - .06543817}} = .03272346 +; \\ &\text{hence } P_{192} = 6.28290510 + \end{aligned}$$

$$\begin{aligned} S_{384} &= \sqrt{2 - \sqrt{4 - .03272346}} = .01636228 +; \\ &\text{hence } P_{384} = 6.28311544 + \end{aligned}$$

$$\begin{aligned} S_{768} &= \sqrt{2 - \sqrt{4 - .01636228}} = .00818121 +; \\ &\text{hence } P_{768} = 6.28316941 + \end{aligned}$$

It is plain that there has been no change to four places of decimals between the perimeters of the polygons of 384 and 768 sides, hence in taking these values for the circumference of the circle there will occur no error in the first four places of decimals, which is sufficiently accurate for ordinary purposes.

Hence, $\pi = \frac{1}{2} C = \frac{1}{2} (6.28316941 +) = 3.14158 +$
or $\pi = 3.1416$ approximately.

EXERCISE V.

Areas.

1. Show that the medians of a triangle divide it into six equivalent triangles.

2. The area of a triangle equals one-half the product of its perimeter by the radius of the inscribed circle.

3. The triangle formed by drawing a line from any vertex of a parallelogram to the middle of the opposite side is equivalent to $\frac{1}{4}$ of the parallelogram.

4. The line joining the middle points of two adjacent sides of a parallelogram forms with them a triangle equal to $\frac{1}{8}$ of the parallelogram.

5. Construct a square equal to the sum of two given squares.

6. Equal to their difference.

7. Change any given triangle into an equivalent right triangle.

8. Trisect a parallelogram by lines drawn from one vertex.

9. The side of an equilateral triangle is 6. Find its area.

10. Show that the quadrilateral formed by drawing lines from the middle points of two of the sides of a triangle to the middle of the third side is a parallelogram whose area is half that of the triangle.

11. Transform a given square into a rectangle having a given diagonal.

12. Bisect a trapezoid by a line parallel to the base.

13. The sides of a triangle are 8 in., 12 in., and 6 in. Find its area.

14. Find the medians of the triangle in Ex. XIII.

15. The sides of a triangle are 5 in., 12 in., and 13 in.

If a perpendicular is dropped from the angle opposite on the side 13 in., find the segments of this side.

16. The base of an isosceles triangle is 34 ft., and one of the legs is 15 ft. Find the altitude.

17. The parallel sides of a trapezoid are 12 in. and 10 in. respectively, and the other two sides are 9 in. and 7 in. Find the area.

18. Construct $\sqrt{2}$ and $\frac{1}{2}\sqrt{3}$.

19. The area of an equilateral triangle is $9\sqrt{3}$. Find its sides.

20. The diagonal of a quadrilateral is 24 in. and the distance between two lines parallel to this diagonal drawn through the opposite vertices is 10 in. Find the area.

21. The diagonals of a parallelogram are 8 in. and 14 in., and one side is 3 in. Find the area.

22. The sides of a triangle are 11 in., 13 in., and 15 in. Find the radius of the circumscribed circle.

23. Two sides of a parallelogram are 8 ft. and 6 ft., and one diagonal is 14 ft. Find the other diagonal.

24. Two triangles of equal area have one angle in each equal and the sides including this angle in one area are 12 in. and 9 in.; one of the including sides in the other is 6 in. Find the remaining including side.

25. A chord 10 in. long is 12 in. from the centre of a circle. What is the area of the circle?

26. The radius of a circle circumscribed about a regular polygon is 10 in.; the side of a polygon is 8 in. What is the area of the polygon?

27. Two tangents are drawn from a point to a circle of radius 6 in.; the tangents are 12 in. long. How far is the point from the centre of the circle?

28. Tangents 22 in. long are drawn to circle hav-

ing a 10 in. radius. Find the length of the chord of contact.

29. The sides of a triangle are 21 ft., 18 ft., and 14 ft. Find its area and from it the three altitudes.

30. The area of a trapezoid is 2,150 sq. yds. One of its parallel sides is 250 yds., and its altitude is 27 ft. What is the other parallel side?

31. One diagonal of a rhombus is $\frac{3}{4}$ the other, and their difference is 6 ft. What is the area?

32. A circle whose radius is 6 in. is inscribed in a quadrilateral whose perimeter is 100 ft. Find the area of the quadrilateral.

33. The sides of a triangle are in the ratio 2 : 3 : 4, and its area is $108\sqrt{15}$. Find the sides.

34. A circular walk is 242 ft. long and 1 yd. wide. How many square yards of cement does it contain?

35. Find the area of a regular hexagon inscribed in a circle whose diameter is 10 in.

36. A wheel of $12\frac{1}{4}$ in. radius has a hole 3 in. in diameter bored in it for a shaft. What is the area of the ring so left?

37. Circles, each with a radius of 7 ft., are drawn with the four vertices of a square as centres and tangent to each other. What is the area of a circle tangent to them all?

38. The circumference of a circle is 33 in. What is the side of an equilateral triangle of the same area [$\pi = 3\frac{1}{7}$]?

SOLID GEOMETRY

DEFINITIONS.

1. A *Surface* has extent in more than one direction but no thickness. As such, it is a pure abstraction and cannot be concretely represented alone. Give a concrete example of a surface?

2. A *Plane* is such a surface that if any two of its points be joined by a straight line the line will lie entirely in the surface. Give an example of a plane?

3. A plane is said to be *determined* by a stated condition, when no other plane can fulfill that condition.

Cor. I. A plane cannot be determined by one straight line. (Why?)

Cor. II. A plane can be determined in four ways:

By three points not in a straight line; by a line and a point without it; by two intersecting straight lines, and by two parallel straight lines. All four, however, practically reduce to the first, for two of the points may be joined to furnish the second condition; or straight lines may be drawn through any two of the points, intersecting in the third; or a straight line may be drawn through any one parallel to a straight line joining the other two.

NOTE. — A plane is unlimited in extent, but from Definition No. 2 it is clear that any qualities possessed by a part of the plane will be characteristic of the whole, hence it is customary, in order to concentrate and fix attention, to represent planes by parallelograms or rectangles.

4. A plane is said to be passed through given points or lines when it is made to contain them.

5. A line is said to be perpendicular to a plane when it is perpendicular to every (or any) line of the plane through its foot.

How would you denote a line neither perpendicular nor parallel to a plane?

NOTE. — The point where a perpendicular pierces the plane is called its foot.

6. A line is said to be parallel to a plane if the line and plane cannot meet, however far produced. Give a definition for parallel planes.

7. The *intersection* of two or more planes is the locus of all points common to the intersecting planes.

8. Points or lines lying in the same plane are said to be *coplanar*. Points lying on the same straight line are called collinear.

Proposition I.

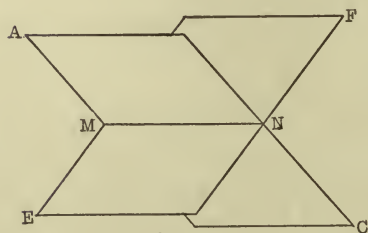


Fig. 1.

The intersection of two planes is always a straight line.

Analysis: The definition of a plane (Art. 2) suggests a method of proof, since if we take any two points in the intersection and join

them by a straight line, the definition (Art. 7) of intersection applies.

Concrete statement from figure:

Let the plane AC intersect the plane EF. Then their intersection is a straight line.

NOTE. — Observe that AC and EF are very small sections of the whole plane.

Proof: Take any two points in the intersection, say M and N. Join these points by a straight line. Since these are points of the intersection, they lie in both planes at the same time.

$\therefore$ the line joining them lies wholly in both planes. (Why?)

$\therefore$ it is their intersection. (Why?)

$\therefore$ their intersection is a straight line.

Comment: There are three general methods of demonstration: the direct method, the indirect (like above proof), and the reduction to an absurdity (*reductio ad absurdum*). The first method is a straight forward progression from hypothesis to conclusion. The second is the assumption that certain known lines are drawn according to the prescribed conditions, and these lines proved identical with the given lines. The third is the assumption of the contrary to the proposition, and the proof that this assumption leads to a geometric absurdity. The next proposition will illustrate the first method.

Proposition II.

If a straight line is perpendicular to two straight lines at their intersection point it is perpendicular to their plane.

Analysis: The definition says that a perpendicular to a plane must be perpendicular to *every* line in the plane drawn through its foot. It is necessary then to prove that the line that is perpendicular to two intersecting lines at their intersection, is also perpendicular to any other line drawn through that point.

Any line means, essentially, *every* line. If any other line is drawn through the foot of this perpendicular, to prove it is also perpendicular, probably the simplest process is to show that it makes a right angle with it. To show that it makes a right angle, since a triangle is easily constructed, of which the perpendicular and this random line are sides, it is suggested to prove this triangle a right triangle. What is the essential relation between the sides of a right triangle?

Statement: Let CD and EF be the intersecting lines in the plane xy , and AB the perpendicular to them at B , their intersection, then is AB perpendicular to xy .

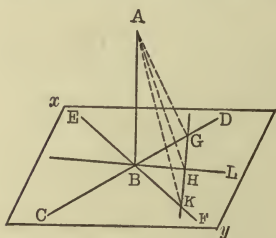


Fig. 2.

Proof: Draw *any* line BL through B . We are to show that AB is $\perp$ to BL .

To include these lines in triangles and to make these triangles as useful as possible, draw GK , cutting CD at G and EF at K , and intersecting BL in such a way

that the intersection point H bisects GK .

(Show how to draw GK to fulfill this condition.) Join A with G , H and K . Then AH is the median of the triangle KAG , and BH is the median of KBG .

$$\text{Hence, } \overline{AK}^2 + \overline{AG}^2 = 2 \overline{AH}^2 + 2 \overline{HK}^2 \quad (1)$$

[by Plane Geometry, 152.]

$$\text{and } \overline{BK}^2 + \overline{BG}^2 = 2 \overline{BH}^2 + 2 \overline{HK}^2 \quad (2)$$

It will be observed that in these two equations are involved two of the sides (hypotenuse and a leg) of the two right triangles ABK and ABG , so that if (2) be subtracted

point to plane MN. Through B draw *any* line CD in the plane, and through B pass a plane RS $\perp$ to CD.

In the plane RS draw AB $\perp$ to FK, the intersection of RS and MN; AB is the $\perp$ required. For CD is $\perp$ to plane RS by construction, hence CD is $\perp$ to AB a line in RS through foot of CD; that is, reciprocally, AB is $\perp$ to CD. But AB was drawn $\perp$ to FK which intersects CD at B and lies in the same plane. $\therefore$ by Prop. II, AB is $\perp$ to plane MN of FK and CD. Draw any other line, say EB, through B to show that it is not $\perp$ to MN.

Since EB and AB intersect in B, they determine a plane; that is, they lie in the same plane (this plane may be RS or some other, depending upon the way EB is drawn). Now if EB is $\perp$ to MN through B, it is $\perp$ to CD. But AB is already proved $\perp$ to CD, which would give us two $\perp$'s to CD at the same point lying in the same plane, but by plane geometry that is impossible: therefore EB is not $\perp$ to MN, and since it, being any line but AB, represents every line through B, except AB, there is no other $\perp$ to MN through B.

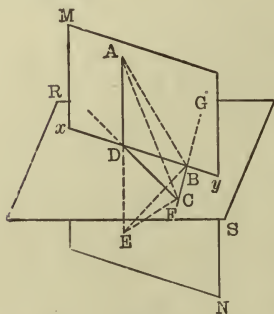


Fig. 4.

draw the plane MN $\perp$ to FG; it will intersect RS in xy. From A drop a $\perp$ AD in MN to xy. It is the $\perp$ required.

If point is without the plane, the proof must be somewhat altered, as there is not now a definite point *in* the plane through which to draw the random line.

Statement: From A without the plane RS to draw a $\perp$ and to show that it is the only one from A to RS. In RS draw any line FG, and through A

Proof: Through D draw any line DC intersecting FG at C. Prolong AD to E, making DE equal to AD. Connect A and E with C and B, the points in which FG intersects DC and xy respectively.

In the triangles ABC and EBC, $\angle ABC$ and $\angle EBC$ are right $\sphericalangle$ s, for AB and BE are in the plane RS to which FG is $\perp$ by construction, hence, FG is $\perp$ to AB and BE (definition). Also BC is common to both triangles and $AB = BE$, because xy is $\perp$ to AE at its middle point D. $\therefore$ the two right triangles are equal and $AC = EC$. $\therefore$ DC is $\perp$ to AE at its middle point, that is, AE or a part of AE, AD is $\perp$ DC, and it was drawn $\perp$ to xy or DB. $\therefore$ AD is $\perp$ to the plane of xy and DC, that is, to RS.

Now to prove that it is the only $\perp$, the analysis is the same as before, and the proof practically the same. Give it.

Proposition IV.

Oblique lines drawn from any point of a perpendicular to a plane, meeting it at equal distances from the foot of the perpendicular, are equal; of two oblique lines meeting the plane at unequal distances from the foot of the perpendicular the more distant is the greater.

Statement: Let P be a point on the $\perp$ OX, SP, NP and MP oblique lines meeting the plane AC so that $ON = OM$, but $OS > ON$ or OM ; then is $NP = MP$ and $SP > NP$ or MP .

Analysis: It is evident that the three lines, the $\perp$, the

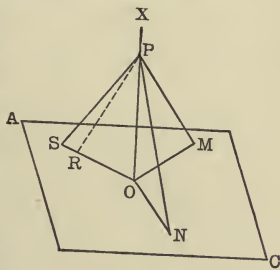


Fig. 5.

oblique, and the distance-lines, as ON, OM and OS, form right triangles. Why right triangles?

It is then only necessary to prove equality or inequality of the triangles to establish our proposition.

Proof (1st part): In the right triangles PON and POM, PO is common and $ON = OM$ (by hypothesis). $\therefore$ triangle PON = triangle POM. $\therefore PN = PM$.

Proof (2d part): On OS lay off $OR = ON = OM$; draw RP. Then by 1st part $RP = NP = MP$. But ORP and OSP being now in the same plane, our demonstration reduces to the Plane Geometry Proposition: If from any point of a line oblique lines are drawn, etc. (complete the statement of the proposition). $\therefore SP > RP$. But $RP = NP = MP$. $\therefore SP > NP$ or MP .

Cor. I. If two equal oblique lines are drawn from any point in a perpendicular to a plane, they meet the plane at equal distances from the foot of the perpendicular.

Cor. II. Since triangles ORP, OMP, and ONP are equal, the $\angle$ s ORP, ONP, and OMP are equal. But $\angle ORP > \angle OSP$ (Why?), and since $SP > RP$, the oblique line having the less inclination to a plane is the greater.

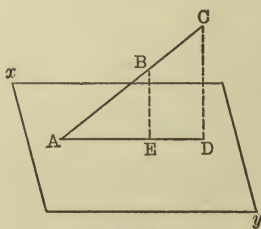


Fig. 6.

Definition: The *projection* of a line on a plane is the line of the plane included between the feet of the perpendiculars dropped from each extremity of the line. If the line touches the plane, its projection is the distance in the plane, from the point of intersection to the foot of the perpendicular dropped from the other end of the line, thus, ED is the projection of BC on plane xy , and AD is the projection of AC on same plane, A being intersection point of AC and xy .

Proposition V.

The locus of points in space equidistant from the extremities of a straight line is the plane perpendicular to this line at its middle point.

Statement: The locus of points equally distant from the extremities of AB is the plane MN $\perp$ to AB at its middle point O.

Analysis: It is necessary, as in the case of all *loci*, to prove that all points in MN are equally distant from A and B, and also that any point outside of MN is unequally distant.

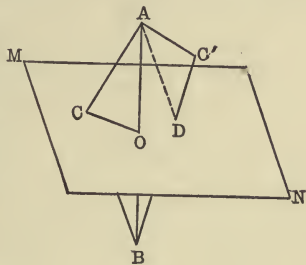


Fig. 7.

Proof: Let C be any point in MN. Join O and C, and C with A and B. Then AOC and OCB are right triangles (Why?), having OC common and $AO = OB$ by construction, hence triangle AOC = triangle OCB, and $\therefore CB = AC$, that is, C is equally distant from A and B.

2d: Let C' be any point not in MN. Join C' with A and B, also D (where BC' cuts the plane) with A.

$AD = DB$ by first part.

$C'D + AD > AC'$, or, since $AD = DB$, $C'D + DB > AC'$, that is, $BC' > AC'$.

C' being any point outside of the plane MN, every point outside is unequally distant.

Proposition VI.

All straight lines perpendicular to a given straight line at a given point lie in a plane perpendicular to the given line at the given point. How else may this proposition be stated?

Statement: Let BC be any perpendicular to EB at B . To show that BC lies in the plane $MN \perp$ to EB at B .

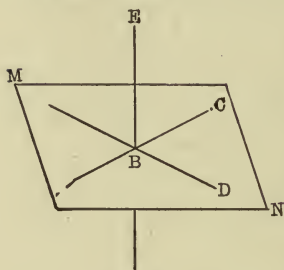


Fig. 8.

Analysis: If it can be shown that such a plane is determined by these perpendiculars the proposition is established.

Proof: Draw any other perpendicular to EB at B , as DB . Then BC and DB , being intersecting lines, determine a plane, say MN . Since EB is $\perp$ to both BC and BD at their intersection, it is $\perp$ to their plane MN . Hence, inversely, MN is $\perp$ to EB at B .

BC and DB being *any* lines represent every line, hence MN is the locus of lines $\perp$ to EB at B . Is this complete?

Proposition VII.

Two straight lines perpendicular to the same plane are parallel.

Statement: Let AB and CD be $\perp$ to MN , then they are $\parallel$.

Analysis: To be parallel they must clearly lie in one plane, and if in that plane they are both $\perp$ to the same straight line, they are $\parallel$.

Proof: Join B and D , the feet of the two perpendiculars. Through D draw $xy \perp$ to BD ; also lay off $FD = ED$, and connect A with F , D and E , and B with F and E .

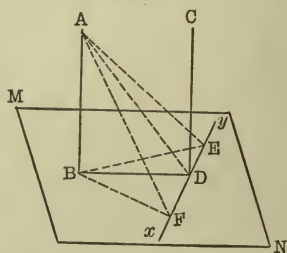


Fig. 9.

$BF = BE$ (Why?), hence $AE = AF$ (Why?). $\therefore AD$ is $\perp$ to xy at D (Why?). Hence EF is $\perp$ to AD and BD at

D; it is also $\perp$ to CD, for CD was drawn $\perp$ to plane which contains xy . But AB lies in the same plane with AD and BD (Why?), and since AD and BD determine a plane with CD, $\therefore$ AB and CD are in the same plane, and are both $\perp$ to the line BD (Why?), hence by Plane Geometry, they are $\parallel$.

Cor. I. If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane. That is, if AB and CD are $\parallel$, and AB is perpendicular to MN, CD is also $\perp$ to MN. Prove it.

Cor. II. If two straight lines are parallel to a third straight line in space, they are parallel to each other.

Let AB and CD be $\parallel$ to EF, then AB and CD are $\parallel$ to each other. For, pass a plane $xy \perp$ to EF, then by Cor. I, AB and CD are both $\perp$ to xy , hence by Prop. VII they are $\parallel$ to each other.

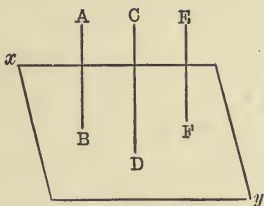


Fig. 10.

Proposition VIII.

If two straight lines are parallel, every plane passed through one of them and only one is parallel to the other.

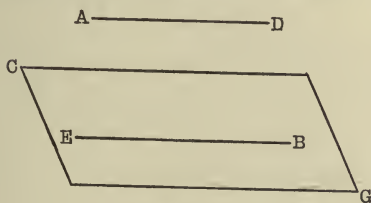


Fig. 11.

Statement: AD is $\parallel$ to EB. The plane CG through EB is $\parallel$ to AD.

Analysis: The proof plainly depends upon the definition of parallelism. It remains to determine whether the

line AD and plane CD can meet, and if so, how?

Proof: AD and EB are in the same plane (Why?), hence

if AD meets the plane CD it must meet it somewhere in EB; but AD cannot meet EB at all, for they are $\parallel$, $\therefore$ AD cannot meet CD.

Cor. I. To pass a plane through either one of two lines in space, parallel to the other. For instance, to pass a plane through CD parallel to AB. Through any point of CD as E draw a line EF $\parallel$ to AB, then the plane determined by CD and EF will be the required plane (Why?).

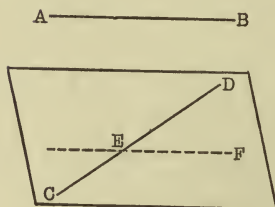


Fig. 12.

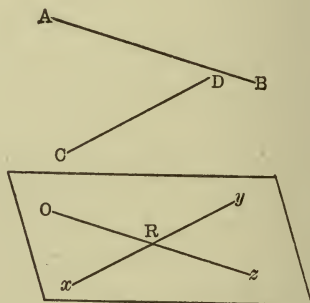


Fig. 13.

Cor. II. To pass through a given point a plane parallel to two given lines in space. To pass a plane through the point R parallel to AB and CD.

Through R draw $Oz \parallel$ to AB, and $xy \parallel$ to CD. The plane determined by xy and Oz is the required plane (Why?).

Proposition IX.

If a straight line is parallel to any line in a plane, it is parallel to the plane.

Statement : Let B be a line in the plane of which MN is a

portion, and let the line A be parallel to B , then is A parallel to the plane of MN .

Analysis: To prove parallelism we must show an impossibility of meeting. We know the lines A and B to be parallel, and if we can make the meeting of A and the plane depend upon the meeting of A and B we can establish our theorem.

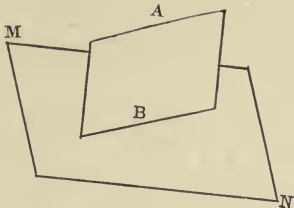


Fig. 14.

Proof: A and B are in the same plane (Why?), and hence if A meets the plane of MN , it must meet the line B (the lines being sufficiently produced). But A and B are parallel by hypothesis, and hence cannot meet, $\therefore A$ cannot meet the plane of MN .

Cor. I. If a line is parallel to a plane, the intersection of that plane with a plane through the line is parallel to the line.

Proposition X.

The intersection made by two parallel planes with a third plane are parallel straight lines.

Statement: If the two parallel planes AB and CD are cut by the plane RS intersecting AB in xy and CD in vz , then xy is $\parallel$ to vz .

Analysis: Parallelism is determined by the non-intersection of the lines which must be proved to lie in the same plane.

Proof: XY and VZ lie in the same plane RS, by hypoth-

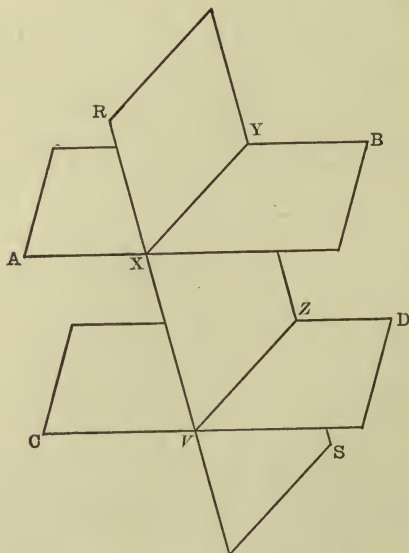


Fig. 15.

esis, and also in the planes AB and CD because they are the intersection of these planes with RS.

Since AB and CD are $\parallel$ by hypothesis, they cannot meet,

and hence no lines in one can meet any lines in the other. Hence, being in the same plane RS and unable to meet, XY is $\parallel$ to VZ.

Cor. Parallel lines included between parallel planes are equal.

Proposition XI.

If two intersecting straight lines are both parallel to a plane, the plane determined by these

lines is parallel to that plane.

Statement: If NP and LM are two intersecting lines, both $\parallel$ to the plane xy , then RS determined by NP and LM is $\parallel$ to xy .

Analysis: Since there is no opportunity of testing directly the possibility of the meeting of xy and RS, we must establish their parallelism by their relation to some line or plane, preferably a perpendicular; hence,

Proof: Draw the line $OB \perp$ to plane xy from the point O (Why from O?). Through OB and NP and OB and LM pass planes PK and LH, intersecting xy in KF and GH. Then GH is $\parallel$ to LM and KF is $\parallel$ to NP (Prop. 9, Cor. I). OB is $\perp$ to KF and GH at B (Why?), hence OB is $\perp$ to NP and LM at O (Why?); $\therefore$ OB is $\perp$ to plane RS (Why?). If, then, both planes, RS and xy , are $\perp$ to OB and could intersect, lines drawn in them from O and B to any point of their intersection would both be $\perp$ to OB. This is impossible, hence RS and xy are parallel.

Cor. It was proved in Prop. XI that OB, which was drawn $\perp$ to plane xy , was also $\perp$ to RS, which was finally proved $\parallel$ to xy , hence, the proof being entirely general, it

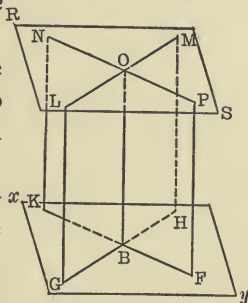


Fig. 16.

follows, that if a line be $\perp$ to one of two $\parallel$ planes it is $\perp$ to the other also.

Proposition XII.

If two angles, although in different planes, have their sides parallel, each to each, they are equal, and their planes are parallel.

Statement: If the sides AR and AS of angle A are $\parallel$ to sides DH and DK of angle D respectively, then $\angle A = \angle D$, and plane xy is $\parallel$ to plane MN.

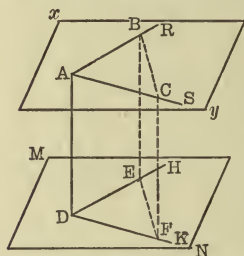


Fig. 17.

Analysis: Where angles are not adjacent or immediately associated with each other, it is necessary, in order to compare them, to make them parts of closed figures (preferably triangles) which can be compared.

As there are no limits set for the sides of these angles, we are free to lay off any distance on them we choose. Clearly, it is advisable to make these distances equal.

Proof: Lay off AB (on AR) and DE (on DH) so that $AB = DE$; also AC (on AS) = DF (on DK), join B and C, and E and F. Draw AD, BE and CF, joining A and D, B and E, and C and F (Why?). Since AC is equal and parallel to DF, by hypothesis, ACFD is a parallelogram; for same reason ADEB is a $\square$, hence AD is equal and parallel to CF and BE, hence they are equal and parallel to each other (Why?). Therefore, BCFE is a $\square$ (Why?), and $\therefore BC = EF$. Hence the triangles ABC and DEF have three sides equal, each to each, $\therefore$ the triangles are equal and $\angle A = \angle D$.

Since two intersecting lines determine a plane, and AR is $\parallel$ to DH, and AS is $\parallel$ to DK, plane xy is $\parallel$ to plane MN.

EXERCISE I.

1. Show how to erect a rod perpendicular to a plane surface.
2. Prove that only one plane can be drawn through a given straight line parallel to another given straight line.
3. Draw a straight line parallel to a given straight line, and meeting each of two other straight lines not situated in the same plane.
4. Draw a straight line of given length parallel to a given plane, and having its extremities on two given straight lines not in the same plane.
5. Find the locus of lines through a given point parallel to a given plane.
6. Find the locus of points in a plane equidistant from two points not in the plane (two cases).
7. Find the locus of points in space equidistant from three points not in a straight line.
8. Find a point in a given plane such that the sum of its distances from two points on the same side of the plane shall be a minimum.
9. What is the locus of points in space equally distant from the vertices of a triangle?
10. If a plane is passed through a diagonal of a parallelogram, the perpendiculars upon this plane from the ends of the other diagonal are equal.
11. Through a point in a plane draw a line in the plane parallel to a given line outside the plane.

Proposition XIII.

If two straight lines are cut by three parallel planes, their corresponding segments are proportional.

Statement: If the two straight lines xy and vz are cut by

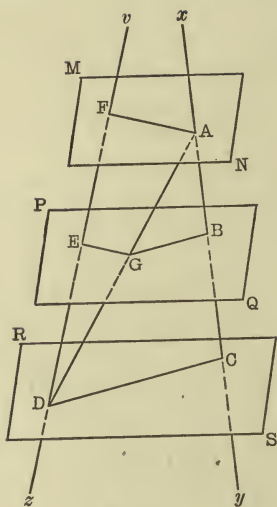


Fig. 18.

the three planes NM , PQ and RS at the points A , B and C (on xy) and F , E and D (on vz), then $AB : BC :: FE : ED$.

Analysis: As, in general, it is necessary to join the two lines in some way, say their extreme points of intersection with the planes MN and RS by a straight line, which will plainly give us two pairs of intersecting lines (AD and AC , and AD and FD), to determine two planes, which will intersect the three parallel planes in parallel lines.

Proof: Join A and D by a straight line, then through AC

and AD pass a plane cutting PQ in BG and RS in CD , also through AD and FD a plane, cutting PQ in GE and MN in AF . In the triangle ACD , BG is $\parallel$ to CD (Why?).

$$\therefore AB : BC :: AG : GD. \quad (1)$$

In the triangle DAF , GE is $\parallel$ to AF . (Why?)

$$\therefore DG : GA :: DE : EF, \text{ or (by inversion) } AG : GD :: FE : ED. \quad (2)$$

Combining (1) and (2); $AB : BC :: FE : ED$.

Definitions.

The space between two intersecting planes is called a *dihedral angle*. The amount of divergence of the two planes determines the size of the angle.

The two planes are called the *faces* of the dihedral angle, and their intersection its edge.

Dihedral angles are called *adjacent, right, oblique*, etc., under the same circumstances as plane angles, planes replacing lines in the definitions.

Dihedral angles are estimated and compared by their plane angles, which are formed by drawing, in each face, straight lines perpendicular at the same point to the edge.

Many of the propositions referring to plane angles apply directly to dihedral angles; as,

(a) If two planes intersect, either one forms with the other two adjacent angles, whose sum is two right dihedral angles.

(b) If two planes intersect, the vertical dihedral angles are equal.

(c) If two parallel planes are cut by a transversal plane, the alternate angles and the interior-exterior angles on the same side of the transversal are equal; also the two interior dihedral angles on the same side of the transversal are supplementary, etc.

Proposition XIV.

Two dihedral angles are equal if their plane angles are equal.

Statement: The two dihedral angles BC and FH (dihedral angles are designated by their edges when standing alone; thus, A-BC-D, E-FH-G, if not) are equal if the plane angles xyz and 123 are equal.

Analysis: Here, as in Plane Geometry, the final test of equality is coincidence, when one figure is applied to another.

Proof: Apply the edge

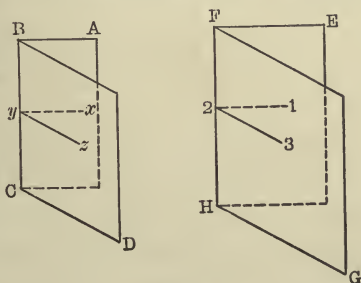


Fig. 19

of FH to the edge of BC , so that the point 2 falls on point y , and the plane EH falls on plane AC ; then 12 will fall on yz , and since $\angle xyz = \angle 123$ (by hypothesis) 23 will fall on yz ; also FH coinciding with BC (two intersecting lines determine a plane), plane FG coincides with plane BD , hence angle BC coincides throughout with FH .

Proposition XV.

Two dihedral angles have the same ratio as their plane angle.

Statement: The two dihedral angles AB and MN have the plane angles CAG and OMX .

Analysis: There are clearly two cases: one, when the two angles have a common measure, which they contain, each, an integral number of times; another, when they have no common measure, that is, are what is called incommensurable.

In the first case, we can compare them by comparing the number of times each takes the common measure.

In the second case, we can use the same method, if we take a unit of measure so small that it leaves an insignificant remainder.

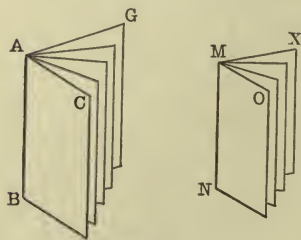


Fig. 20.

Proof: Case I. Divide the angles CAG and OMX into parts each equal to the unit of measure that they will contain; say it is contained in CAG , m times, and in OMX ,

n times; then $CAG : OMX :: m : n$. (1)

Through these points of division and the edges AB and MN pass planes; these planes will divide the dihedral

angles AB and MN respectively into m and n equal dihedral angles; hence, $\angle AB : \angle MN :: m : n$. (2)

Combining proportions (1) and (2) having equal couplets, $\angle AB : \angle MN :: \angle CAG : \angle OMX$.

Proof: Case II.

Statement: $\angle BK : \angle MT :: \angle ABD : \angle OMP$ when the angles are incommensurable.

Proof: If we take an angle unit exactly contained in $\angle OMP$, say, and divide $\angle ABD$ by that unit, it will be contained in $\angle ABD$ a certain number of times with a remainder FBD .

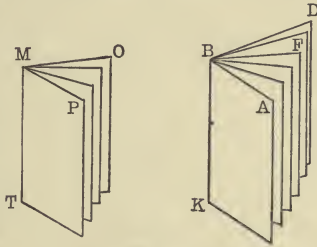


Fig. 21.

Planes passed through these points of division will divide the dihedral angles into corresponding parts, $\angle MT$ into an integral number of parts, and BK into a number of parts with a remainder $F-BK-D$, which of course will be less than the unit used.

Then the part $A-BK-F$ of BK is commensurable with MT , and hence, by part 1, $\angle A-BK-F : \angle MT :: \angle ABF : \angle OMP$.

Now by taking a smaller unit, say an aliquot part of the original unit, the $\angle MT$ will still contain an integral number of parts; and the remainder in BK , since it must always be less than the dividing unit, can be made smaller and smaller, as we use a smaller and smaller unit. Clearly, there is no limit to the decrease in the remainder, hence we can make the remainder as nearly zero as we please; but wherever the division point F is situated, the proportion $\angle A-BK-F : \angle MT :: \angle ABF : \angle OMP$ is always true, even when the point F is practically coincident with the point D ; hence, in general, $\angle BK : \angle MT :: \angle ABD : \angle OMP$.

Proposition XVI.

If two planes are perpendicular to each other, any line drawn in one of them, perpendicular to their intersection, is perpendicular to the other plane.

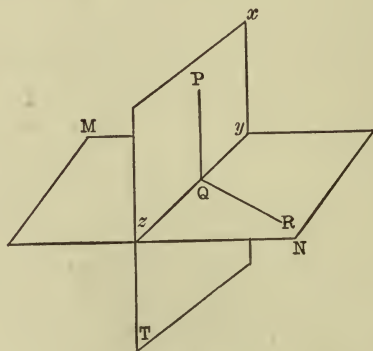


Fig. 22.

Statement : If the planes MN and xT are $\perp$ to each other, the line PQ drawn in xT , $\perp$ to zy , the intersection of the planes is $\perp$ to MN .

Analysis: Since the plane angle of a dihedral measures it and in this case PQ is one side of such angle, it is plainly indicated to complete this angle. Hence,

Proof: In MN draw $QR \perp$ to zy at Q . Then since PQ and QR are both $\perp$ to zy by construction, $\angle PQR$ is the plane angle of the dihedral having zy for its edge, but this dihedral angle is a right angle by hypothesis. $\therefore \angle PQR$ is a right angle; that is, PQ is $\perp$ to QR , but it is also $\perp$ to zy ; $\therefore$ it is $\perp$ to MN . (Why?)

EXERCISE.

Prove that if two planes are perpendicular to each other, a perpendicular to one of them at a point in their intersection will lie in the other.

Suggestion: Draw a $\perp$ in the other plane and prove its identity with the given $\perp$.

Cor. If two planes are perpendicular to each other, a perpendicular to one of them from any point in the other will lie in that other plane.

Proposition XVII.

If a straight line is perpendicular to a plane, every plane passed through that line will be perpendicular to the plane.

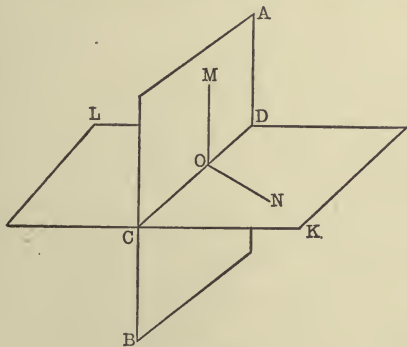


Fig. 23.

Statement: If the line MO is $\perp$ to plane LK , intersecting it at O , and the plane AB be any plane passed through MO AB is $\perp$ to LK .

Analysis: Construction of plane angle, evidently.

Proof: In LK draw to the point O the line $ON \perp$ to the intersection CD of the two planes. Then is $\angle MON$ the plane angle of the dihedral $A-CD-K$. (Why?)

But $\angle MON$ is a right angle (Why?), $\therefore$ the dihedral CD is a right dihedral, and hence AB is $\perp$ to LK .

Cor. A plane perpendicular to the edge of a dihedral angle is perpendicular to both its faces.

Proposition XVIII.

If two intersecting planes are both perpendicular to a third plane, their intersection is perpendicular to that plane.

If the two planes CD and AB intersecting in EF are both $\perp$ to GH , then EF is $\perp$ to GH .

Analysis: If we can identify a $\perp$ to GH at F with the intersection of CD and AB , our theorem is proved.

Proof: By the exercise under Prop. XVI, if a $\perp$ be

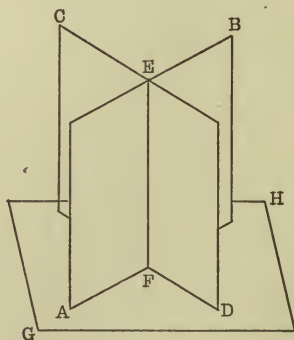


Fig. 24.

erected to GH at the point F (common to both AB and CD) it will lie in both AB and CD , that is, it will be their intersection and identical with EF , hence EF is a $\perp$ to GH .

Cor. If three planes are mutually perpendicular, the intersection of any two is perpendicular to the third.

Proposition XIX.

Every point in a plane which bisects a dihedral angle is equally distant from its faces.

Statement: Every point in the plane AC bisecting the dihedral angle formed by the planes AN and AM, is equally distant from those planes.

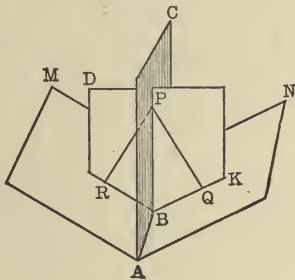


Fig. 25.

Analysis: This proposition suggests a Plane Geometry proposition, and if we can reduce the demonstration to a single plane, we can make use of it.

Proof: Take any point P in AC and draw the perpendiculars PQ and PR to AN and AM respectively. Then we are to prove $PQ = PR$.

Since PQ and PR intersect at P, they determine a plane DK, which is $\perp$ to both AM and AN (Why?), hence it is perpendicular to their intersection AB, or AB is $\perp$ to plane DK.

$\therefore$ AB is $\perp$ to BR, BQ and BP (the intersection of DK and AC) (Why?). Hence, $\angle PBR$ and $\angle PBQ$ are the plane angles respectively, of the two dihedrals formed by AC and AM, and by AC and AN. Therefore, since these two dihedrals are equal, being halves of the same

angle, $\angle PBR = \angle PBQ$. In the two right triangles PBQ and PBR, $\angle PBQ = \angle PBR$, and $PB = PB$ (common side).

$\therefore$ being right triangles they are equal throughout and $PQ = PR$.

Definitions.

The *projection of a point* on a plane is the point where the perpendicular upon that plane from the point cuts it.

The *projection of a line* on a plane is the locus of the projection of all its points on that plane. The projection of a perpendicular to a plane on that plane is a point.

The projection of a straight line that does not cut the plane, is a straight line joining the feet of the perpendiculars dropped from each end of the line to the plane. If the line cuts the plane, its projection is the straight line joining its point of intersection in the plane, with the foot of the perpendicular dropped from the end (or ends) of the line upon the plane. The angle which a line makes with a plane is the angle it makes with its projection on the plane.

Proposition XX.

Through a given straight line not perpendicular to a plane, one plane can be passed perpendicular to the given plane.

Statement: If AB is not perpendicular to the plane EF, one plane can be passed through AB $\perp$ to EF, say AD, intersecting EF in CG.

Analysis: If any point in AB be selected and a perpendicular drawn from it to the plane, any plane passed through this perpendicular will be $\perp$ to the plane EF by a previous

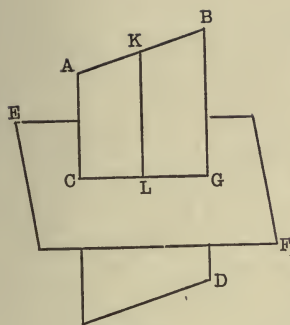


Fig. 26.

proposition; it then remains to show that there is but one such plane.

Proof: From K, any point in AB, drop the perpendicular KL to the plane EF. Through this perpendicular and AB pass the plane AD intersecting EF in CG. This plane will be $\perp$ to EF. It is the only plane through AB which would be perpendicu-

lar; because if two planes could be passed through AB $\perp$ to EF, AB would be, of course, their intersection and would be $\perp$ to EF, but by hypothesis AB is *not* $\perp$ to EF. Hence but one such plane can be drawn.

Proposition XXI.

The acute angle which a straight line makes with its projection on a plane is the least angle it makes with the plane.

Statement: Let BC be the projection of AB on the plane MN, then is $\angle ABC$ less than any other angle made with the plane, say $\angle ABD$.

Analysis: To compare angles or lines, whenever possible, it is desirable to include them in triangles whose parts may be compared.

Proof: Draw $BD = BC$ and AD (AC is the $\perp$ determining the projection of AB on MN).

In the two triangles ABC and ABD, AB is common and

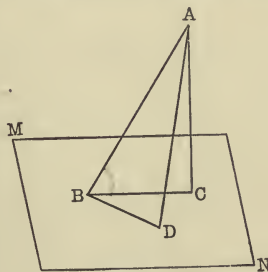


Fig. 27.

$BC = BD$ (by construction); but $AC < AD$ (Why?). Hence $\angle ABC < \angle ABD$, where ABD represents any angle made by AB and MN .

Proposition XXII.

Between two straight lines not in the same plane, there can be only one common perpendicular.

Statement: If PR and CD are two lines not in the same plane, they can have only one common perpendicular.

Analysis: The lines must be compared through some

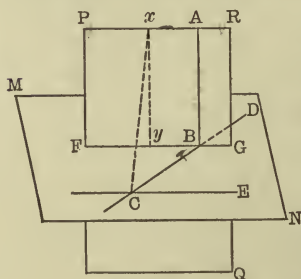


Fig. 28.

connecting plane, which shall bear a known relation to a plane determined by one of them, and it is clearly desirable that these planes be $\perp$.

Proof: Through any point as C of CD draw a line CE parallel to PR , then the plane MN , determined by CE and CD , is parallel to PR (Why?).

Through PR pass a plane PQ perpendicular to MN intersecting it in FG . It will intersect CD , say at B , otherwise CD would be $\parallel$ to PR , contrary to hypothesis, which says they are *not* in the same plane. At B erect a $\perp$ AB to MN ; it will lie in PQ by a previous proposition.

Now AB is $\perp$ to CD and FG (Why?), but FG is $\parallel$ to PR (Why?), hence AB is $\perp$ to PR , since PR and FG lie in the same plane and are $\parallel$. Hence AB is $\perp$ to both PR and CD . If it is not the only perpendicular, let xC (drawn from any other point x in PR to C) be $\perp$ to both PR and CD . Then xC is $\perp$ to CD and CE (Why?), and hence $\perp$

to MN. From x draw $xy \perp$ to FG, then xy is $\perp$ to MN (by previous proposition; cite it), and there would then be two $\perp$'s from x to MN, which is impossible.

Definitions.

The space included by three or more planes intersecting in a point is called a *polyhedral angle*.

The point of intersection is called its *vertex* and the planes its *faces*. The lines in which adjacent faces intersect are called its *edges*, and the plane angles between the edges are called *face angles*.

The size of the polyhedral angle depends solely upon the amount of divergence of its faces and not upon their extent nor upon their shape.

Each pair of adjacent faces of a polyhedral angle form a dihedral angle.

A polyhedral angle is convex or concave according as a section cutting all its edges is convex or concave.

A polyhedral angle of three faces is called a *trihedral angle*; of four faces, a *tetrahedral angle*, etc.

A trihedral angle may evidently have one, two, or three right dihedral angles.

If the edges of a polyhedral angle are prolonged through its vertex, another polyhedral angle is thus formed called a *symmetrical polyhedral angle*, with its faces arranged in reverse order.

Proposition XXIII.

The sum of any two face angles of a trihedral angle is greater than the third face angle.

Statement: In the trihedral angle A-KLM, $\angle KAL + \angle LAM > \angle KAM$.

Analysis: To compare these angles they must be as

Proposition XXIV.

The sum of all the face angles of any polyhedral angle is less than four right angles.

Statement: If A is a polyhedral angle, then $\angle BAC + \angle CAD + \angle DAE + \angle EAF + \angle FAG + \angle GAB$ is $<$ four right angles, B, C, D, E, F and G being any points on its edges.

Analysis: To estimate parts of a polyhedral angle, the parts must be definitely enclosed and defined, and that can be done only by passing a plane through the polyhedral cutting all edges.

Proof: Pass a plane through the points B, C, D, E, F and G, thus forming the polygon BCDEFG. Take any point K within this polygon and connect it with the vertices B, C, D, E, F and G, thus forming triangles with the common vertex K.

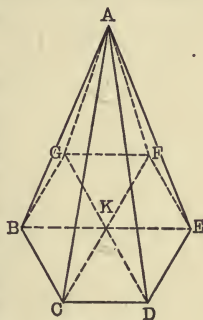


Fig. 30.

At each vertex, B, C, D, etc., are trihedral angles, where by the last proposition, the sum of two face angles is greater than the third. Hence,

$$\begin{aligned} & \angle BCA + \angle ACD > \angle BCD \\ \text{or} \quad & \angle BCA + \angle ACD > \angle BCK + \angle KCD. \\ & \angle ADC + \angle ADE > \angle CDE \\ \text{or} \quad & \angle ADC + \angle ADE > \angle KDC + \angle KDE. \\ & \angle AED + \angle AEF > \angle DEF, \text{ etc.} \end{aligned}$$

That is, the sum of all the angles at the base of the triangles having vertex A is greater than the sum of all the angles at the base of the triangles having vertex K.

But the sum of *all* the angles of the triangles with vertex A is equal to the sum of *all* the angles of triangles with

vertex K, because there are the same number of triangles in each and the sum of the angles of any triangle equals two right angles, hence the sum of the angles at A is less than the sum of the angles at K. This can be represented thus: Call the sum of the base angles of triangles having vertex K, X; and the sum of the base angles of triangles having vertex A, Y; then $X < Y$ (by what has been proved).

Call the sum of *all* the angles of triangles having vertices K and A, each Z (for it is the same in both).

Then

$$Z = Z$$

$$\underline{X < Y.}$$

Subtract; $Z - X > Z - Y$ (the less subtrahend leaves greater remainder).

But $Z - X = \text{sum of angles at K}$

and $Z - Y = \text{sum of angles at A.}$

The sum of the angles at K = 4 right angles. (Why?)

$\therefore$ The sum of the angles at A < 4 right angles.

Proposition XXV.

Two trihedral angles are equal if the three face angles of one are equal to the three face angles of the other.

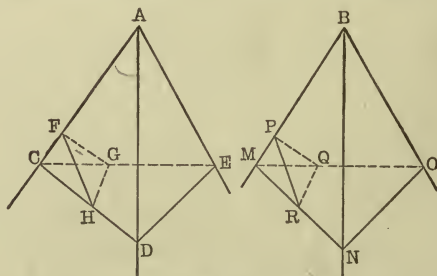


Fig. 31.

Statement: If in the two trihedrals A and B, $\angle CAD =$

$\angle MBN$, $\angle DAE = \angle NBO$, and $\angle CAE = \angle MBO$, then $\angle A = \angle B$.

Analysis: The two trihedrals are evidently equal if their dihedral angles are equal, for then they will exactly coincide, when superposed; and the dihedrals are equal if their plane angles are equal, hence we must make such a construction that the plane angles of two corresponding dihedrals can be compared, and that requires equal triangles including those angles. We are clearly at liberty to construct these as simply as possible.

Proof: Lay off from A and B equal distances on all the edges, say AC, AD, AE, BM, BN and BO. Join C, D and E, also M, N and O.

Then the triangles CAD and MBN, DAE and NBO and CAE and MBO have each two sides and the included angle equal, hence are equal.

Hence $CD = MN$, $DE = NO$ and $CE = MO$.

Also $\angle ACD = \angle BMN$, $\angle ACE = \angle BMO$, etc.

Then triangles CDE and MNO have three sides in each equal, hence they are equal and $\angle ECD = \angle OMN$.

On AC and BM, respectively, lay off $CF = MP$, and at F and P construct the plane angles of the dihedrals AC and BM, HFG and RPQ. The sides of HFG intersect CE in G and CD in H, and the sides of RPQ intersect MO in Q and MN in R. (Why?)

Join G and H, and R and Q by straight lines.

In the *right* triangles, HCF and RMP,

$CF = MP$ (by construction).

$\angle FCH = \angle PMR$ (by above proof).

$\therefore$ triangle HCF = triangle RMP, and hence $FH = PR$, $CH = MR$.

The *right* triangles CFG and MPQ are equal for same reason, hence $FG = PQ$ and $CG = MQ$.

In the triangles (not *right*) HCG and RMQ

$$CH = MR \text{ (by above proof).}$$

$$CG = MQ \text{ (by above proof).}$$

$$\angle HCG = \angle RMQ \text{ (by proof).}$$

$\therefore$ triangle CHG = triangle RMQ and $HG = RQ$.
(Why?)

In triangles HFG and RPQ, $FH = PR$,

$$FG = PQ, \quad HG = RQ.$$

$\therefore$ triangle HFG = triangle RPQ

and $\angle HFG = \angle RPQ$. (Why?)

$\therefore$ dihedral angle AC = dihedral BM.

In the same way all the other dihedrals may be proved equal. It will be observed how all the demonstration is concentrated upon the proving of equality between the parts of triangles HFG and RPQ, and how the construction was made to assist in this aim.

EXERCISE II.

Prove:

1. The planes that bisect two supplementary-adjacent dihedral angles, are perpendicular to each other.
2. Show that a plane perpendicular to the edge of a dihedral angle is perpendicular to both faces.
3. Prove the converse of 2.
4. What is the greatest number of equilateral triangles, squares and pentagons, respectively, that can be grouped about a point to form a polyhedral angle.
5. From two given points on the same side of a given plane, draw two lines to a point in the plane, meeting it at equal angles. What is the locus of the point of meeting in the plane?
6. What is the locus of points in space equally distant from two intersecting lines?

7. Find a point equally distant from four given points not in the same plane.

8. If a line is parallel to one plane and perpendicular to another, the two planes are perpendicular to each other.

9. Find the locus of points in space equally distant from two planes (two cases).

10. Find the locus of points equally distant from two lines not in the same plane.

11. Show that a plane may be drawn perpendicular to only one edge, and to only two faces of a polyhedral angle.

12. If a straight line is parallel to a plane, any plane perpendicular to the line is perpendicular to the plane.

13. If a line is inclined to a plane at an angle of 60° , its projection on the plane is equal to half the line.

GEOMETRICAL SOLIDS.

Definitions.

A *polyhedron* is a solid bounded by planes.

A *diagonal* is a straight line joining any two vertices of a polyhedron not in the same face.

A *section* of a polyhedron is the portion of an intersecting plane inclosed by its intersections with the faces of the polyhedron.

A polyhedron is convex if every section of it is a convex polygon.

A polyhedron of four faces is called a *tetrahedron*. A polyhedron of five faces is called a *pentahedron*. A polyhedron of twenty faces is called an *icosahedron*, etc.

PRISMS AND PARALLELOPIPEDS.

A *prism* is a polyhedron, two faces of which are equal polygons in parallel planes, and the other faces are parallelograms.

The equal polygons are usually called the *bases*; the parallelograms, *lateral faces*; and the intersections of these faces, *lateral edges*.

The sum of the lateral faces is the *lateral area*.

A *right prism* is one whose lateral edges are perpendicular to the planes of the bases. A prism whose edges are not perpendicular to the bases is an *oblique prism*.

A *regular prism* is a *right prism* with regular bases. Prisms are named by their bases, *triangular*, *quadrangular*, etc.

A *parallelopiped* is a prism, whose bases are parallelograms. A *right parallelopiped* is one whose lateral edges are perpendicular to the planes of its bases; if its 6 faces are all rectangles, it is called a *rectangular parallelopiped*.

A *cube* is a parallelopiped whose faces are squares. A cube whose edges are a linear unit in length has been chosen as the *unit of volume*.

The *altitude* of any prism is the perpendicular distance between its bases.

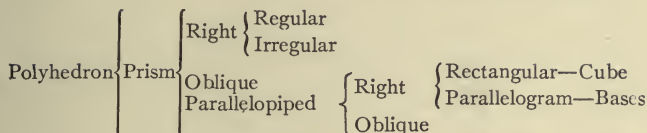
The *volume* of any solid is the number of units of volume which it contains.

Two solids having equal volumes, are said to be *equivalent* (not equal). When would they be equal?

A *right section* of a prism is a section perpendicular to the lateral edges.

A *truncated prism* is a portion of a prism included between its base and any oblique section.

SCHEMATIC ARRANGEMENT.



Proposition XXVI.

The sections of a prism made by parallel planes cutting all the lateral edges are equal.

Statement: In the prism AB, if the sections 12345 and 67890 are $\parallel$ they are equal.

Analysis: Equal figures must be capable of coincidence if superposed. Hence must have all parts equal.

Proof: The sides 12, 23, 34, 45 and 51 are $\parallel$ respectively to 67, 78, 89, 90 and 06 (Why?), and since the edges of the prism are parallel by definition 12, 23, 34, etc., are equal to 67, 78, 89, etc., respectively. The angles 123, 234, 345, 451 and 512 are respectively equal to 678, 789, 890, 906, 067 (Why?). Hence the two polygons having equal sides and angles could be made to coincide.

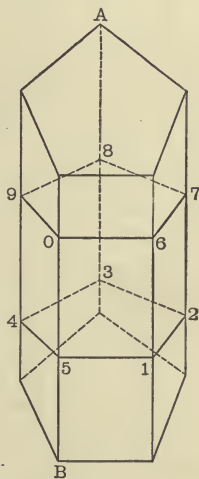


Fig. 32.

Cor. Every section of a prism parallel to the bases is equal to the bases; and all right sections of a prism are equal.

Proposition XXVI (a).

The lateral area of a prism is equal to the perimeter of a right section multiplied by a lateral edge.

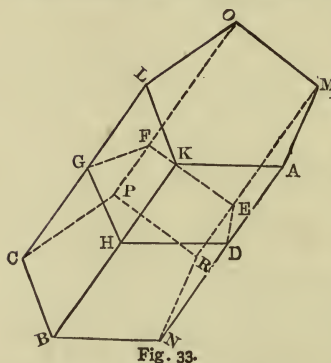


Fig. 33.

Statement: If DEFGH is a right section of the prism AC, then the lateral area = $(DE + EF + FG + GH + HD) \times BK$ (or any other edge).

Analysis: Since the lateral area is made up of parallelograms, and the area of a parallelogram is

equal to the product of base and altitude, it is simply a question of determining the bases and altitudes of these component parallelograms.

Proof: By definition of right section, the sides of the section are all perpendicular to the edges, hence taking the edges as bases of the parallelograms these sides will be their altitudes.

$$\begin{aligned} \text{Hence} \quad \text{face AB} &= KB \times DH, \\ \text{" KC} &= KB \times HG, \\ \text{" LP} &= LC \times GF, \\ \text{" OR} &= OP \times FE, \\ \text{" MN} &= MR \times ED, \end{aligned}$$

and so on for any number of faces. Adding; lateral area = $KB \times DH + KB \times HG + LC \times GF + OP \times FE + MR \times ED = (DH + HG + GF + FE + ED) \times KB$, since all the edges are equal. (Why?)

Proposition XXVII.

Any oblique prism is equivalent to a right prism whose base is equal to a right section of the oblique prism, and whose edges (or altitude) are equal to the edges of the oblique prism.

Statement: If the prism 1-67890 is a right prism having a base 67890 = a right section of the prism A-MNRPQ and an altitude (same as an edge in a right prism) equal to an edge of the prism A-MNRPQ, then are these prisms equivalent.

Analysis: The usual test of geometrical values, superposition for comparison. As in the nature of the case exact coincidence is impossible, the next best thing is to bring the figures together in order to compare parts.

Proof: Combine the two prisms so that a base of the right prism will become a right section of the oblique prism; then the edges of the two prisms will take the same direction (Why?), the edges of the right prism extending beyond the upper base of the oblique prism, so that

$$A_1 = M_6, B_2 = N_7, C_3 = R_8, \text{ etc.} \quad (\text{Why?})$$

The entire solid is made up of a portion of each original prism and a common truncated prism A-67890. An examination of the figure will show that if A-67890 is added to 6-MNRPQ, the result is the oblique prism;

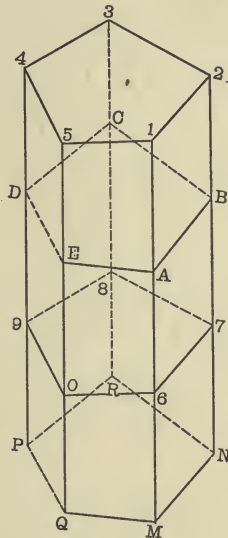


Fig. 34.

and if it is added to $1 - ABCDE$, the result is the right prism, hence if it can be shown that $6 - MNR PQ = 1 - ABCDE$, our proposition is easily established.

As always, superposition will make it possible to compare these two truncated prisms $6 - MNR PQ$ and $1 - ABCDE$. Place the base $MNR PQ$ of one upon $ABCDE$ the base of the other, Q on E ; they will exactly coincide, for they are the bases of the original oblique prism, and hence are equal. The lateral faces, $MQO6$, $QP9O$, $PR89$, etc., are equal respectively to the faces, $AE51$, $ED45$, $DC34$, etc. (Why?)

Hence the edges $M6$, QO , $P9$, etc., will fall upon the edges $A1$, $E5$, $D4$, etc. (since the face angles of the trihedrals A , E , D , etc., are equal, these trihedrals are equal by former Proposition). Since $M6 = A1$, $QO = E5$, $P9 = D4$, etc., 6 will fall on 1 , O on 5 , 9 on 4 , etc., hence the bases $6789O$ and 12345 must coincide,

$$\begin{array}{rcl} & 6 - MNR PQ & = 1 - ABCDE; \\ \text{and} & A - 6789O & = A - 6789O \\ \text{adding;} & A - MNR PQ & = 1 - 6789O. \end{array}$$

Cor. I. Two prisms having the three faces of a trihedral angle in one equal to the three faces (similarly placed) of a trihedral angle in the other are equal.

Cor. II. Two right prisms are equal if they have equal bases and altitudes.

EXERCISE.

Show that the opposite faces of a parallelepiped are equal and parallel.

Proposition XXVIII.

If a plane be passed through two diagonally opposite edges of a parallelopiped, it will divide the parallelopiped into two equivalent triangular prisms.

Statement: Let the plane ED be passed through the edges BE and DG of the parallelopiped $B - EFGH$, then the two triangular prisms so formed, $C - EGH$ and $A - EFG$ are equivalent.

Analysis: Prop. XXVII and Cor. II under it, give us a clue to the comparison of prisms in general, through the medium of right sections, which act as bases for right prisms.

Proof: Since it is necessary to express oblique prisms in terms of right prisms, in order to compare them, Prop. XXVII suggests a method.

Make a right section, 1234 of the parallelopiped; it will cut the plane ED in its diagonal 13 , which will divide it into equivalent triangles, since it will be a parallelogram. (Why?) Then these triangles will be right sections of the two triangular prisms.

The triangular prism $C - EHG$ is equivalent to a right prism, whose base is the right section 143 , and whose altitude is an edge, and $A - EFG$ is equivalent to right prism with base 123 . But $143 = 123$, and the edges are all equal, $\therefore$ these two right prisms will be equivalent, hence the prisms, $C - EHG$ and $A - EFG$ are equivalent.

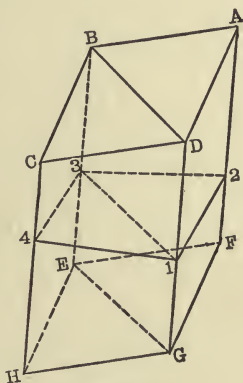


Fig. 35.

Proposition XXIX.

Two rectangular parallelopipeds having equal bases are to each other as their altitudes.

Statement: If the bases of the two parallelopipeds, D — EFGH and M — QTSR are equal, then D — EFGH : M — QTSR :: DE : MQ.

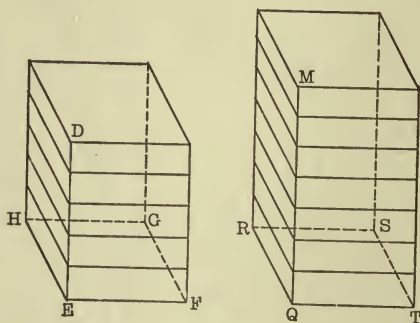


Fig. 36.

Analysis: There are plainly two cases: when the altitudes (edges) are commensurable and when they are incommensurable. In the first case it is necessary only to apply the common measure and in the second case to take the unit of measure so small that we may approach commensurability as near as we please.

Proof: Case I:

Apply the common measure to the altitudes (edges) DE and MQ; say it is contained in DE, x times, and in MQ, y times, then,

$$DE : MQ :: x : y.$$

Through the points of division pass planes $\parallel$ to the bases; they will divide the parallelopipeds into x and y parts, respectively, which will all be equal. (Why?)

Hence, $D - HEFG : M - RQTS :: x : y$.
 and $D - HEFG : M - RQTS :: DE : MQ$.

Case II is left to the student.

Cor. Since the area of a rectangle equals the product of its two dimensions, Prop. XXIX may be stated thus: Two parallelopipeds, having two dimensions equal, are to each other as the third dimension.

Proposition XXX.

Two rectangular parallelopipeds having equal altitudes are to each other as their bases.

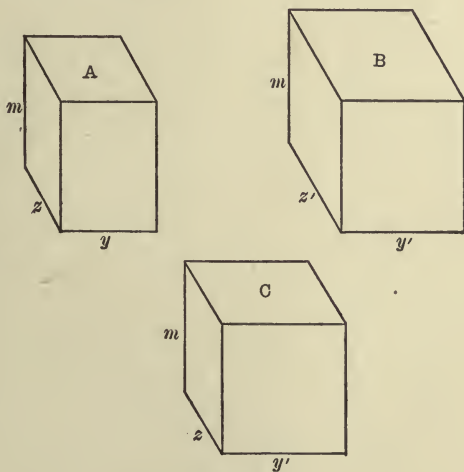


Fig. 37.

Statement: Let the parallelopipeds A and B have the same altitude m and bases $y \times z$ and $y' \times z'$.

$$\frac{A}{B} = \frac{y \times z}{y' \times z'}.$$

Analysis: Since any face of a parallelopiped may be taken as its base, another parallelopiped can be constructed having two dimensions in common with both A and B, with which both may be compared.

Proof: Construct the rectangular parallelopiped C with the dimensions m, y', z .

Then taking $m \times z$ as the base of C, $\frac{A}{C} = \frac{y}{y'}$ (Prop. 29);

and taking $y' \times m$ as base of C, $\frac{C}{B} = \frac{z}{z'}$, (by Prop. 29).

By multiplying these two equations together,

$$\frac{A}{C} \times \frac{C}{B} = \frac{y}{y'} \times \frac{z}{z'} \text{ or } \frac{A}{B} = \frac{y \times z}{y' \times z'}.$$

Cor. Prop. XXX may evidently be stated thus: Two rectangular parallelopipeds having one dimension equal are to each other as the product of the other two.

Proposition XXXI.

Two rectangular parallelopipeds are to each other as the product of their three dimensions.

Statement: If the two parallelopipeds, M and N, have the dimensions a, b, c and a', b', c' , respectively, then

$$\frac{M}{N} = \frac{a \times b \times c}{a' \times b' \times c'}.$$

Analysis: Compare each with another parallelopiped having partly the dimensions of both.

Proof: Construct the parallelopiped R with the dimensions a, b , and c' . Then,

$$\frac{M}{R} = \frac{c}{c'} \text{ and } \frac{R}{N} = \frac{a \times b}{a' \times b'}$$

multiplying these equations,

$$\frac{M}{K} \times \frac{K}{N} = \frac{a \times b \times c}{a' \times b' \times c'} \quad \text{or} \quad \frac{M}{N} = \frac{a \times b \times c}{a' \times b' \times c'}.$$

Cor. The volume of a rectangular parallelopiped equals the product of its three dimensions. For suppose a certain

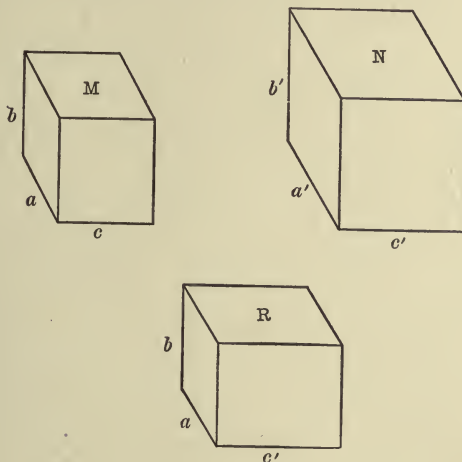


Fig. 38.

parallelopiped A has the dimensions a , b , c , and a cube B has the edges each one unit in length (the same unit in which a , b and c are expressed) or has the dimensions 1, 1, 1. By Prop. XXXI,

$$\frac{A}{B} = \frac{a \times b \times c}{1 \times 1 \times 1} = a \times b \times c.$$

But volume is defined as the number of times a solid contains the unit of volume, which is universally taken to be a cube, whose edges are one unit in length, hence $\frac{A}{B} =$ the volume of A $= a \times b \times c$.

Proposition XXXII.

The volume of any parallelopiped is equal to the product of its base and altitude.

Statement: The oblique parallelopiped D — EGFH = $EGFH \times OP$, OP being its altitude, as indicated in Fig. 39.

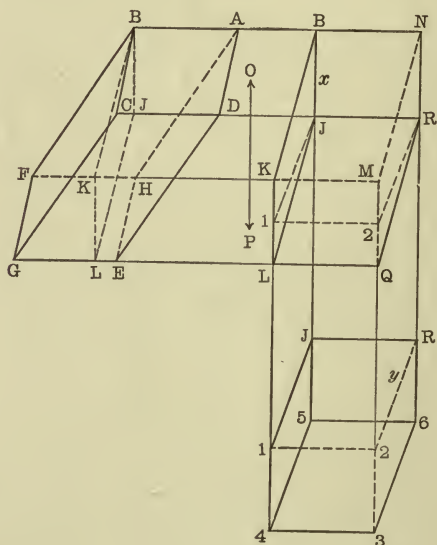


Fig. 39.

Analysis: Since it is known that the volume of a rectangular parallelopiped equals the product of its three dimensions, which is equivalent to saying the product of its base by its altitude, it is necessary to express the oblique parallelopiped in terms of an equivalent rectangular parallelopiped. Also it is known that an oblique parallelopiped is equivalent to a right parallelopiped whose base equals a

right section of the oblique and whose altitude is an edge. If right sections be taken successively perpendicular to each other in the oblique parallelopiped, the result will clearly be a rectangular parallelopiped, no matter how oblique the original may have been. It remains only to prove the equivalence of these successive figures.

Proof: Through the vertex B pass the right section BJLK and move it out parallel to its first position along the produced edges BA, CD, FH and GE to x ; then on these produced edges lay off $KM = FH$, $BN = BA$, $LQ = GE$ and $JR = CD$ and draw the equivalent right section MNRQ. B — LQMK is then a right parallelopiped equivalent to D — EGFH, for its base (any face of a parallelopiped may be its base) BJLK is a right section of D — EGFH and its altitude BN is equal to the edge CD.

B — LQMK is not yet rectangular.

Through the edge JR pass the right section JR 21 and move it out to y , along the produced edges. On these produced edges lay off R 6, J 5, 14, and 23 equal to the edges BJ, etc., and draw the equivalent right section 3456 forming the rectangular parallelopiped J — 1234.

Now these three parallelopipeds all have the same altitude OP, because the planes have remained unchanged in direction; also the bases 1234, KMQL and EGFH are all equal, because the sides 23 and MQ and LQ and GE are equal by construction, and the opposite sides lie in parallel lines, hence their altitudes are the same.

But $J - 1234 = \text{base} \times \text{alt.} = 1234 \times OP$,
and since the parallelopipeds are all equivalent,

$$D - EGFH = J - 1234 = 1234 \times OP = \\ EGFH \times OP;$$

that is, $D - EGFH = EGFH \times OP$.

Proposition XXXIII.

The volume of a triangular prism is equal to the product of its base by its altitude.

Statement: The volume of the triangular prism C — FGH = $\text{FGH} \times \text{OP}$.

Analysis: It is clearly necessary to express the triangular prism in terms of a parallelopiped, since we have determined the volume of the latter. The relation between a triangle and a parallelogram of the same base and altitude, will suggest a method of converting the triangular prism into a parallelopiped.

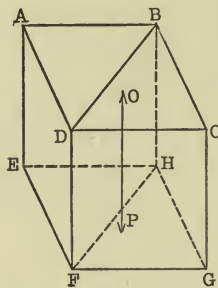


Fig. 40.

Proof: Complete the parallelograms of which the triangular bases of the prisms are halves.

Thus, draw $AB \parallel$ to DC ; $AD \parallel$ to BC ; $EF \parallel$ to HG and $EH \parallel$ to FG in the planes of the triangles BDC and FGH .

Join A to E , and we have a parallelopiped $A - EFGH$, whose bases are twice the bases of the prism (Why?), and which has the same altitude, OP .

The volume of this parallelopiped = $\text{EFGH} \times \text{OP}$.

But $\text{EFGH} = 2 \text{FGH}$ and by Prop. XXVIII

$\text{Vol. A} - \text{EFGH} = 2 \text{ vol. C} - \text{FGH}$.

$\therefore \text{Vol. A} - \text{EFGH} = \text{EFGH} \times \text{OP}$ can be expressed thus:

$$2 \text{ Vol. C} - \text{FGH} = 2 \text{ FGH} \times \text{OP}.$$

$$\therefore \text{Vol. C} - \text{FGH} = \text{FGH} \times \text{OP}.$$

Proposition XXXIV.

The volume of any prism equals the product of base by altitude.

Statement: The volume of the prism $E - FGHL$ = the base $FGHKL$ multiplied by the altitude M .

Analysis: Since every prism may be divided into triangular prisms by passing planes through the non-adjacent edges, the characteristics of triangular prisms belong to any prism.

Proof: Through the edges EL and CH , also EL and BG , pass planes dividing the prism into the triangular prisms $A - FGL$, $E - GLH$ and $D - LKH$, having the same altitude M , as the original prism, and the bases FGL , GLH and LKH make up the base, $FLKHG$, of the original prism. Also the original prism equals the sum of the triangular prisms.

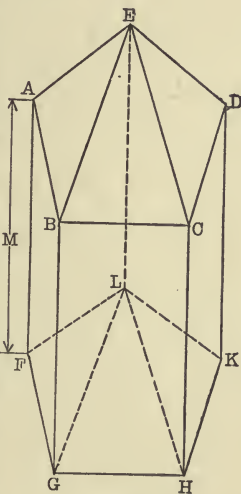


Fig. 41.

Now $A - FLG = FLG \times M$; $E - GLH =$

$GLH \times M$ and $D - LKH = LKH \times M$

and $E - FLKHG = A - FLG + E - GLH + D - LKH$,
also $FGHKL = FGL + GLH + LKH$.

Hence, $E - FLKHG = (FLG + GLH + LKH) \times M$
or $E - FLKHG = FGHKL \times M$.

Definitions.

A *pyramid* is a polyhedron whose base is a polygon and whose lateral faces are triangles having a common vertex. Draw one. The intersections of the faces are

called *edges*, as usual; and the common vertex is known as the *apex*, or the *vertex*, of the pyramid. The sum of the lateral faces is called the *lateral area*.

The *altitude* of a pyramid is the perpendicular distance from the apex to the plane of the base.

Pyramids are named according to their bases; *triangular*, *quadrangular*, *pentagonal*, etc.

A triangular pyramid is also called a tetrahedron, since it has four faces.

A pyramid whose base is a regular polygon with its centre at the foot of the perpendicular from the apex to the base, is called a *regular* pyramid. This perpendicular from the apex to the base-centre is called the *axis* of the regular pyramid.

The lateral faces of a regular pyramid are equal isosceles triangles (Why?).

The *slant height* of a regular pyramid is the altitude of any one of these isosceles triangles.

Would an oblique pyramid have a *slant height*?

A *truncated pyramid* is the portion of a pyramid included between the base and any section made by a plane cutting all lateral edges. When the section is parallel to the base, the truncated pyramid is called a *frustum* of a pyramid.

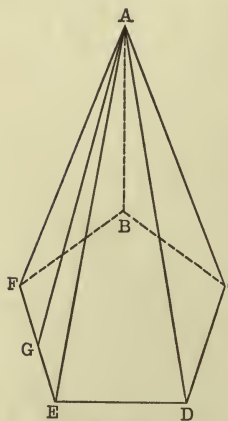


Fig. 42.

Proposition XXXV.

The lateral area of a regular pyramid is equal to half the product of the slant height and the perimeter of the base.

Statement: If $A - BCDEF$ is a regular pyramid, whose slant height is AG then its lateral area is

$$\frac{1}{2} (BCDEF \times AG).$$

Analysis: By definition of regular pyramid and slant height, the relation between the slant height and the triangular faces is evident.

Proof: Each face being an isosceles triangle, all having the same altitude; namely, the slant height of the pyramid, its area will equal $\frac{1}{2}$ its base $\times$ slant height, and its base will be one side of the base of the pyramid. That is,

$$ABC = \frac{1}{2} BC \times AG;$$

$$ACD = \frac{1}{2} CD \times AG;$$

$$ADE = \frac{1}{2} DE \times AG; \text{ etc.}$$

Adding; lateral area $= \frac{1}{2} (BC + CD + DE + EF + FB) \times AG$.

Cor. The lateral area of the frustum of a regular pyramid equals one-half the sum of the perimeters of the bases multiplied by the slant height.

What is the shape of the faces of the frustum?

Proposition XXXVI.

A section parallel to the base of a pyramid is similar to the base and divides the edges and altitude proportionally.

Statement: Let $MNOPQ$ be a section of the pyramid $A - BCDEF$ $\parallel$ to $BCDEF$ cutting the altitude at R . Then $AM : AB :: AN : AC :: AO : AD :: AR : AT$, etc.

Analysis: Parallel planes cut by transversal planes suggest the relations of the lines of intersection in the triangular faces, and hence the

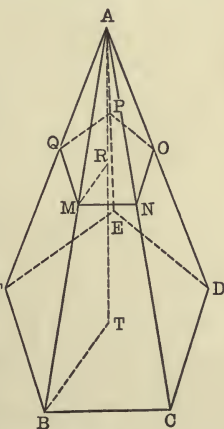


Fig. 43.

relations between the section-polygon and the base-polygon. (Why?)

Proof:

Since MN is $\parallel$ to BC , NO $\parallel$ to CD , OP $\parallel$ to DE , etc.

$$AM : AB :: AN : AC$$

Also $AN : AC :: AO : AD$

$$AO : AD :: AP : AE, \text{ etc.}$$

Hence,

$AM : AB :: AN : AC :: AO : AD :: AP : AE$, etc.,
and in the right $\triangle$'s AMR and ABT ,

$$AM : AB :: AR : AT. \quad (\text{Why?})$$

2d Part:

Since MN is $\parallel$ to BC , NO $\parallel$ to CD , PO $\parallel$ to ED , etc.

$$\angle MNO = \angle BCD, \angle NOP = \angle CDE, \text{ etc.} \quad (\text{Why?})$$

Also $MN : BC :: AN : AC$

and $NO : CD :: AN : AC$, etc.

$$\therefore MN : BC :: NO : CD, \text{ etc.}$$

Hence, $MNOPQ$ and $BCDEF$ are equiangular and have proportional sides, $\therefore$ are similar.

Cor. I. Any section of a pyramid parallel to the base, is to the base as the square of its distance from the vertex is to the square of the altitude.

$$\text{That is, } MNOPQ : BCDEF :: \overline{AR}^2 : \overline{AT}^2.$$

For $MNOPQ$ and $BCDEF$ being similar polygons,

$$MN : BC :: AM : AB :: AR : AT$$

$$\text{or } \overline{MN}^2 : \overline{BC}^2 :: \overline{AR}^2 : \overline{AT}^2.$$

$$\therefore MNOPQ : BCDEF :: \overline{AR}^2 : \overline{AT}^2.$$

Cor. II. If two pyramids having equal altitudes are cut by planes $\parallel$ to the bases and at equal distances from the apices, the sections have the same ratio as the bases. Prove it.

Cor. III. If in *Cor. II* the pyramids have also equal bases, the sections are equal.

Proposition XXXVII.

Two triangular pyramids having equivalent bases and equal altitudes are equivalent.

Statement: If the two pyramids $A - BCD$ and $A' -$

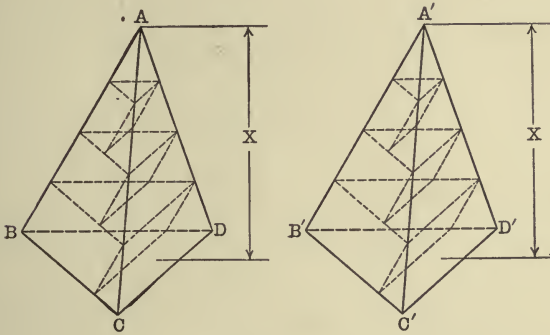


Fig. 44.

$B'C'D'$ have equal bases (BCD and $B'C'D'$) and equal altitudes X , they are equivalent.

Analysis: By inscribing small equal prisms in each pyramid the volumes may be compared, if the prisms be made so small that their sum is practically equal to the pyramids.

Proof: Divide the altitude into y equal parts and through the points of division pass planes parallel to the bases in both pyramids.

Also, through the intersections of these planes with any one face (in both pyramids), pass planes $\parallel$ to the opposite lateral edge.

There will be formed in both pyramids a series of prisms, each prism in one being equal to its correspond-

ing prism in the other; hence the two series of prisms are equal.

It is apparent that by making the number of parts into which the altitude is divided, sufficiently small, and hence each prism sufficiently small, the aggregate of the prisms in each pyramid can be made as nearly equivalent to the volume of the pyramids as we please. But the series are always equivalent no matter how small the individual prisms, hence the pyramids are equivalent.

Proposition XXXVIII.

The volume of a triangular pyramid is equal to one-third the product of its base and altitude.

Statement: The volume of the pyramid B — DEF, having base DEF and altitude, $H = \frac{1}{3} DEF \times H$.

Analysis: If we can express a relation between pyramids and prisms we can find the value of this pyramid for we know the value of prisms.

Proof: Through the vertices D and F of the base of the pyramid draw CF and AD parallel and equal to BE. Through A, B and C pass a plane; also planes through edges AD and BE, also CF and BE, thus constructing a prism on the base DEF, with an altitude equal to that of the pyramid B — DEF.

This prism is composed of the triangular pyramid, B — DEF, and the quadrangular pyramid B — ADFC. Through A, B and F pass a plane, forming triangular pyramids B — ADF and B — AFC.

Since ACFD is a parallelogram, the diagonal AF (the line in which the plane ABF intersects the face DFCA) divides this parallelogram into two equivalent triangles, which are the bases of the two component triangular prisms.

These triangular prisms have the equivalent triangles ADF and AFC for bases, and equal altitudes, because they have the same apex, and their bases in the same plane, hence they are equivalent.

Also the pyramids $B - DEF$ and $F - ABC$, have equal bases (the bases of the prism) and equal altitudes, hence they are equivalent.

Now $F - ABC$ may be read $B - ACF$ (taking ACF as the base and B as apex, since any face of a triangular pyramid may serve as its base).

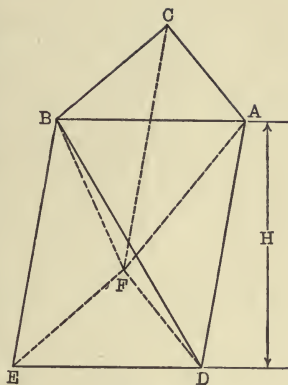


Fig. 45.

$$B - DEF = F - ABC = B - ACF.$$

$$\therefore B - ADF = B - ACF = B - DEF. \quad (\text{Axiom.})$$

Hence the prism is divided into three equivalent pyramids, of which $B - DEF$ is one.

$$\begin{aligned} \text{Hence,} \quad & B - DEF = \frac{1}{3} \text{ the prism } ABC - DEF \\ \text{and} \quad & ABC - DEF = DEF \times H \text{ (the altitude).} \\ \therefore & B - DEF = \frac{1}{3} (DEF \times H), \end{aligned}$$

and since DEF is the base of $B - DEF$, and H is its altitude, the proposition is proved.

Proposition XXXIX.

The volume of any pyramid is equal to one-third the product of its base and altitude.

Statement: Let $A - BCDEF$ be a pyramid with base

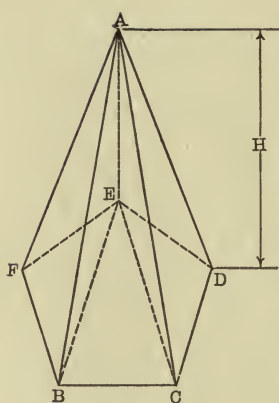


Fig. 46.

BCDEF and altitude H, then the volume of ABCDEF = $BCDEF \times H$.

Analysis: By passing planes through the edges and diagonals of the base the pyramid is easily divided into triangular pyramids of equal altitude, that are equal to $\frac{1}{3}$ the product of the base and altitude. Complete the proof as in Prop. XXXIII.

Cor. I. Pyramids are to each other as the product of bases and altitudes, and if their

bases are equal they are to each other as their altitudes; if their altitudes are equal they are to each other as their bases.

Cor. II. The volume of any polyhedron may be found by dividing it into triangular pyramids.

Proposition XL.

The volume of a frustum of a pyramid is equal to one-third its altitude times the sum of the areas of its two bases and their mean proportional.

Statement: Let A — BCDEF be a frustum of altitude x and bases y and z ; then its volume = $\frac{1}{3}x(y + z + \sqrt{y.z})$.

Analysis: A frustum of a pyramid is evidently the difference between two pyramids, one having the lower base of the frustum as base, and the other, the upper base, and having the same apex A of the pyramid from which the frustum was cut.

Call the volume of the pyramid A — MNOPQ, v , and

the volume of the pyramid
A - BCDEF, V.

Let the altitude of the pyramid A - MNOPQ be T, then
volume of A - BCDEF =
 $\frac{1}{3} (x + T) y$, and volume of A -
MNOPQ = $\frac{1}{3} T \cdot z$.

Then volume of the frustum
N - BCDEF = $\frac{1}{3} (x + T) y$
- $\frac{1}{3} T \cdot z = \frac{1}{3} x \cdot y +$
 $\frac{1}{3} T (y - z)$ (by rearrange-
ment).

$$\frac{y}{z} = \frac{(x + T)^2}{T^2} .$$

(Prop. XXXVI, Cor. I.)

$$\therefore \frac{\sqrt{y}}{\sqrt{z}} = \frac{x + T}{T} ,$$

and

$$T \sqrt{y} = x \sqrt{z} + T \sqrt{z} ,$$

or

$$T (\sqrt{y} - \sqrt{z}) = x \sqrt{z} ,$$

hence,

$$T = \frac{x \sqrt{z}}{\sqrt{y} - \sqrt{z}} .$$

Substituting this value of T in the expression for volume of frustum,

$$\begin{aligned} \frac{1}{3} (x \cdot y) + \frac{1}{3} T (y - z) &= \frac{1}{3} x \cdot y + \frac{1}{3} \frac{x \sqrt{z}}{\sqrt{y} - \sqrt{z}} (y - z) \\ &= \frac{1}{3} x \cdot y + \frac{1}{3} x \sqrt{z} (\sqrt{y} + \sqrt{z}) = \frac{1}{3} x (y + z + \sqrt{y \cdot z}) . \end{aligned}$$

Cor. Since $\frac{1}{3} x \cdot y$, $\frac{1}{3} x \cdot z$ and $\frac{1}{3} x \sqrt{y \cdot z}$ are each the volume of a pyramid with the common altitude x and bases y , z and $\sqrt{y \cdot z}$, respectively, how else may proposition be stated? Divide the frustum into these three pyramids.

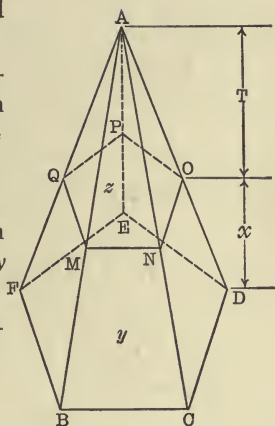


Fig. 47.

Proposition XLI.

Two triangular pyramids having a trihedral angle in each equal, are to each other as the products of the edges including the equal trihedral angles.

Statement: If $A' - BCD$ and $A - EFG$ are two triangular prisms having the trihedral angles A and A' equal, then

$$\frac{A' - BCD}{A - EFG} = \frac{A'B \times A'D \times A'C}{AE \times AF \times AG}.$$

Analysis: As usual when two geometrical magnitudes are to be compared, especially when they are known to have equal parts, superposition is the logical process.

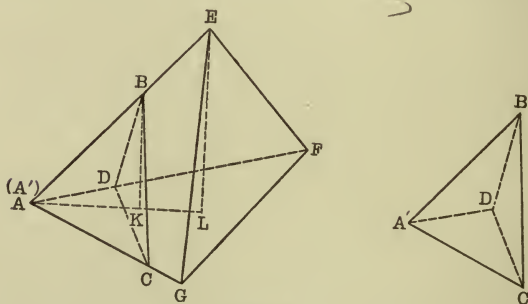


Fig. 48.

Proof: Combine the two pyramids, making A' coincide with A , as to edges and faces, thus $A' - BCD$ will take the position $A - BCD$ within $A - EFG$.

From B and E drop perpendiculars BK and EL to the face AFG , and through BK , EL and A pass a plane, intersecting the face AFG in AKL . Taking ACD as the base of $A - BCD$ and AFG as the base of $A - EFG$, BK and

EL will be their respective altitudes, and by Prop. XXXIX, Cor. I.

$$\frac{A - BCD}{A - EFG} = \frac{\frac{1}{3} ACD \times BK}{\frac{1}{3} AFG \times EL} = \frac{ACD \times BK}{AFG \times EL}.$$

But $\frac{ACD}{AFG} = \frac{AC \times AD}{AG \times AF}$ (Why?)

and $\frac{BK}{EL} = \frac{AB}{AE}$, (Why?)

hence

$$\frac{A - BCD}{A - EFG} = \frac{ACD \times BK}{AFG \times EL} = \frac{AC \times AD \times AB}{AG \times AF \times AE}.$$

Cor. Two tetrahedrons are similar if three faces of one are similar respectively to three faces of the other, and are similarly placed.

Definition: Similar polyhedrons are those that have the same number of faces, respectively similar and placed in the same order, and their polyhedral angles equal.

It follows easily from this definition that similar polyhedrons may be decomposed into the same number of similar tetrahedrons; also that the homologous edges of similar polyhedrons are proportional, and that any two homologous lines have the same ratio as two homologous edges.

From the theory of proportion, it follows, since the surface of similar polyhedrons are made up of similar polygons, that these surfaces are to each other as the squares of any two homologous edges.

Proposition XLII.

The volumes of two similar tetrahedrons are to each other as the cubes of their edges.

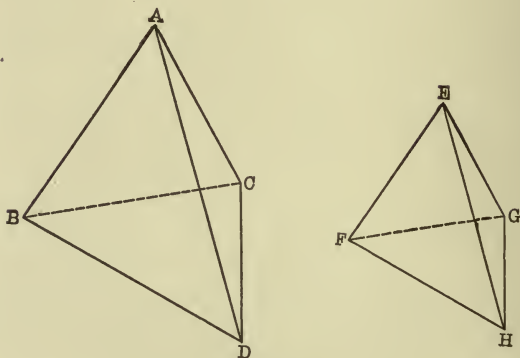


Fig. 49.

Statement: If $A - BCD$ and $E - FGH$ are two similar tetrahedrons, then

$$\frac{A - BCD}{E - FGH} = \frac{\overline{AB}^3}{\overline{EF}^3} \text{ or } \frac{\overline{AC}^3}{\overline{EG}^3} \text{ or } \frac{\overline{AD}^3}{\overline{EH}^3}, \text{ etc.}$$

Let the volume of $A - BCD$ be V , and of $E - FGH$, be V' .

By Prop. XLI,

$$\frac{V}{V'} = \frac{AB \times AC \times AD}{EF \times EG \times EH} = \frac{AB}{EF} \times \frac{AC}{EG} \times \frac{AD}{EH}.$$

Since the faces are respectively similar,

$$\frac{AB}{EF} = \frac{AC}{EG} = \frac{AD}{EH},$$

$$\therefore \frac{V}{V'} = \frac{AB}{EF} \times \frac{AB}{EF} \times \frac{AB}{EF} = \frac{\overline{AB}^3}{\overline{EF}^3} = \frac{\overline{AC}^3}{\overline{EG}^3} = \frac{\overline{AD}^3}{\overline{EH}^3}$$

Proposition XLIII.

The volumes of two similar polyhedrons are to each other as the cubes of any two homologous edges.

Analysis: Similar polyhedrons can be divided into similar tetrahedrons, by planes; and since the polyhedrons are the aggregates of the tetrahedrons, and the tetrahedrons are to each other, in pairs, as the cubes of homologous edges, which are at the same time homologous lines of the similar polyhedrons, the proof of the proposition is evident. Complete it.

EXERCISE III.

1. Any section of a prism made by a plane parallel to an edge is a parallelogram.
2. Show that the diagonals of opposite faces in a parallelepiped are parallel.
3. The diagonals of a parallelepiped bisect each other.
4. In a rectangular parallelepiped the sum of the squares of the four diagonals equals the sum of the squares of the twelve edges.
5. The volume of any triangular prism is equal to half the product of any lateral face by the perpendicular to this face from the opposite edge.
6. The volume of any prism is equal to the product of its lateral area, by half the apothem of the base.
7. The square of the diagonal of a cube equals three times the square of its edge.
8. If from any point within a regular tetrahedron perpendiculars be dropped upon the four faces, the sum of these perpendiculars equals the altitude of the tetrahedron.
9. Prove that a line drawn through the point of intersection of the diagonals of a parallelepiped and terminating in opposite faces, is bisected at the point.

10. If the four diagonals of a quadrangular prism pass through a common point the prism is a parallelepiped.

11. The base of a prism is a right triangle, whose longest and shortest sides are respectively 17 and 8 in. ; the height of the prism is 13 in. What is its volume?

12. The base of a right prism is a quadrilateral, whose sides are 8, 10, 12, and 14 in. and the lateral edges of the prism are 21 in. Find its lateral area.

13. The sides of the base of a right triangular prism are 24, 28, and 32 and its height is 10. Find its total area.

14. Two rectangular parallelepipeds have bases whose dimensions are 8×15 and 9×24 respectively. If their altitudes are equal what is the ratio of their volumes?

15. The dimensions of two rectangular parallelepipeds are respectively 5, 9, 16, and 9, 12; the ratio between their volumes is 1:3. What is the third dimension of the second?

16. Find the dimensions of a cube equal in volume to a parallelepiped whose dimensions are 27, 28, and 35.

17. If the dimensions of a brick are $8 \times 4 \times 3$ in. how many bricks will it require for a wall 8 in. thick, 30 ft. long and 20 ft. high?

18. A stack has the form of a frustrum of a cone. The diameter of its lower base is 6 ft. and of its upper base 4 ft. If it is 50 ft. high, what will it cost to paint it at 7 cents per square yard?

19. What will it cost to excavate a trench 4 ft. deep, 6 ft. wide at the top, 4 ft. wide at the bottom, and 50 ft. long, at 16 cents per cubic yard?

20. Taking 231 cu. in. to the gallon, how much water will a bucket hold, if it is 12 in. high, has a bottom diameter of 8 in., and a top diameter of $12\frac{1}{2}$ in.?

21. How much slate will it take to cover a tower in the form of a regular quadrangular pyramid, 24 ft. high and with a base 10 ft. on a side, if the slate averages 12×8 in. and 6 in. is uncovered?

22. A section of a tunnel is a rectangle 8×12 ft., surmounted by a semi-circle whose diameter is the smaller dimension of the rectangle. If the tunnel is $\frac{3}{4}$ of a mile long, how many cubic yards of dirt were removed?

23. A log is 15 ft. long, 16 in. diameter at the larger end and 12 in. diameter at the smaller. What will be the dimensions of the largest square section pillar that can be sawed from it, and how many cubic feet of waste will there be?

24. The areas of the bases of a frustum of a pyramid are 48 and 27 sq. ft., respectively, and its altitude is 9 ft. What is the altitude of the completed pyramid?

25. The altitude and lateral edge of a frustum of a regular triangular pyramid are 4 and 5, respectively, and each side of its upper base is $\sqrt{3}$. Find its volume.

26. The bases of frustum of a pyramid are rectangles whose sides are 27 and 15, and 9 and 5, respectively, and the line joining their centres is perpendicular to both and equal to 12. Find lateral area and volume.

27. The lateral surface of a pyramid is greater than its base.

28. The altitude of a pyramid is divided into three equal parts by planes parallel to its base? Find the ratio between the three solids thus formed.

29. The volume of a cube is $1\frac{4}{9}$ cu. ft. Find the length of its diagonals.

30. The lines joining each vertex of a tetrahedron with the point of intersection of the medial lines of the opposite faces all meet in a common point, which divides each line

in the ratio 1 : 4 (this point being the centre of gravity of the tetrahedron). Prove it.

31. What is the edge of a cube whose entire surface is 63.375 square feet?

32. On three given lines intersecting in space, construct a parallelopiped.

33. The altitude of a pyramid is 64 in. Divide the pyramid into three equal parts by planes parallel to the base, by determining the points on the altitude, where the planes cut them.

34. A section of a tetrahedron made by a plane parallel to two non-intersecting edges is a parallelogram. Prove it.

35. The altitude of a pyramid is 8 ft. and its base is a regular hexagon with 4 foot sides. Find volume.

36. The slant height of the frustrum of a regular pyramid is 15 in.; the sides of its square bases are respectively 30 and 12 in. Find volume of frustrum.

37. Within a 40-foot steeple, in the form of a quadrangular pyramid with square base, a platform is to be built 25 ft. from its base. If the base of the steeple is 16 ft. on a side, how many square feet of lumber will be required for the platform?

38. Around the same steeple at a height of 28 ft. a platform 3 ft. wide is to be built. How many square feet of lumber will be required?

39. A wagon bed is 5×17 ft. at the top, and 4×16 ft. at bottom, corresponding dimensions. If it is 4 ft. deep, how many bushel of coal, counting 2,688 cubic inches to the bushel, will it hold?

40. A square chimney is 120 ft. 9 in. high; its width at the base is 10 ft. 9 in., and at the top, 5 ft. 9 in. The cavity is a square prism 2 ft. 6 in. on the side. How many cubic feet of masonry in the chimney?

CYLINDERS.

Def. I. A *cylindrical surface* is one generated by a line moving parallel to a fixed straight line and always touching a fixed curve, not in the same plane as the fixed straight line.

The moving line is called the *generatrix*, and the fixed curve is called the *directrix*.

The generatrix in any position is called an *element* of the cylindrical surface.

Def. II. A *cylinder* is a solid bounded by a cylindrical surface with a *closed* directrix and two parallel sections of it. Compare with a prism.

The parallel sections are called the *bases*, and the cylindrical surface, the *lateral surface*.

Def. III. The *altitude* of a cylinder is the perpendicular distance between its bases.

Def. IV. A *right* cylinder is one whose elements are all perpendicular to the planes of the bases. In other words it is one whose elements are all equal to the altitude.

Def. V. A *circular* cylinder is one whose bases are circles. What must be the form of the directrix?

Def. VI. It is evident that a right circular cylinder may be generated by a rectangle, revolving about one of its sides as an axis. Such a cylinder is often called a *cylinder of revolution*.

By the word *cylinder* is usually meant a circular cylinder.

Def. VII. The line joining the centres of the bases is called the *axis* of a cylinder.

Def. VIII. A plane is *tangent* to a cylinder, when it contains only one element.

Def. IX. A prism is said to be inscribed in a cylinder, when its bases are inscribed in the bases of the cylinder,

and its lateral edges are elements of the cylinder. Define circumscribed prisms.

Remarks: It is perfectly clear that the bases of a cylinder are equal; that any section parallel to the bases is equal to them; and that sections that are parallel and cut all the elements are equal.

Proposition XLIV.

Any section of a cylinder made by a plane passing through an element is a parallelogram.

Statement: If $abcd$ is such a section of the cylinder ad , it is a $\square$.

Analysis: The definition of a $\square$ requires a proof that the sides of $abdc$ are $\parallel$.

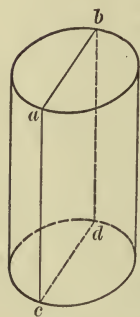


Fig. 50.

Proof: Let bd be the element through which the section is drawn. Through a draw an element of the cylinder; this element lies in the plane $abdc$ (Why?), and, being an element, also lies in the surface of the cylinder, hence is the intersection of

$abdc$ with the surface, hence it coincides with ac . But it is parallel to bd , since they are both elements.

$\therefore ac$ is $\parallel$ to bd . Also bd is $\parallel$ to cd . (Why?)

$\therefore abdc$ is a $\square$.

Proposition XLV.

The lateral area of a circular cylinder is equal to the product of the circumference of a right section by an element.

Statement: Let cd be a right section and ab an element of the cylinder ae , then the lateral area $S = \text{circumference of } cd \times ab$, say, $S = C \times ab$.

Analysis: The proposition immediately suggests the corresponding proposition on the prism, and as it is readily

shown that a cylinder is nothing but a prism with an infinite number of faces, we are led to inscribe a prism and increase its faces indefinitely.

Proof: Inscribe a regular prism of any number of faces in the cylinder; call its lateral area, S' , its lateral edge, ab , and the perimeter of its right section inscribed in the right section of the cylinder, P . Then $S' = P \times ab$.

But if the number of faces of this prism be indefinitely increased, say by successive doubling of the number of sides, it will ultimately coincide with the cylinder. Then

$$S' = S \text{ and } P = C.$$

$$\therefore S = C \times ab.$$

Remark: Similarly it may be proved that the volume of a cylinder equals the product of its base and altitude. In fact all the propositions applying to prisms apply equally to cylinders.

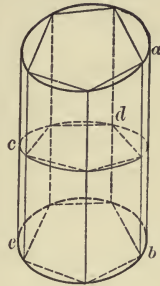


Fig. 51.

Proposition XLVI.

The lateral or entire areas of similar cylinders of revolution are to each other as the squares of their altitudes (or an element) or as the squares of their radii, and their volumes are to each other as the cubes of their altitudes or radii.

Statement: Let A and B be two cylinders of revolution; S and S' their lateral surfaces; T and T' their total surfaces; V and V' their volumes; R and R' their radii; and H and H' their altitudes, then $S : S' :: R^2 : R'^2 :: H^2 : H'^2$; $T : T' :: R^2 : R'^2 :: H^2 : H'^2$ and

$$V : V' :: R^3 : R'^3 :: H^3 : H'^3.$$

Analysis: Being cylinders of revolution they are generated by rectangles, which, of course, are similar, and since the radii and altitudes constitute the two dimensions of these rectangles their ratios are equal.

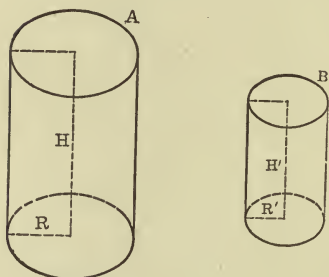


Fig. 52.

Proof: Since the lateral areas of cylinders, like prisms, equal the perimeter of a right section multiplied by an element, and these being right cylinders, the bases are right sections and the altitudes equal the elements,

$$S = 2 \pi R H \text{ and } S' = 2 \pi R' H', \quad T = 2 \pi R^2 + 2 \pi R H, \\ T' = 2 \pi R'^2 + 2 \pi R' H', \quad V = \pi R^2 H \text{ and } V' = \pi R'^2 H'.$$

[From like propositions on prisms.]

(The total area includes the two bases with the lateral surface and the volume equals the product of base by altitude.)

$$\frac{S}{S'} = \frac{2 \pi R H}{2 \pi R' H'} = \frac{R H}{R' H'} = \frac{R}{R'} \times \frac{H}{H'},$$

and since the rectangles are similar

$$\frac{R}{R'} = \frac{H}{H'} \text{ hence } \frac{S}{S'} = \frac{R}{R'} \times \frac{H}{H'} = \frac{R}{R'} \times \frac{R}{R'} = \frac{R^2}{R'^2},$$

$$\text{or} \quad = \frac{H}{H'} \times \frac{H}{H'} = \frac{H^2}{H'^2}.$$

$$\frac{T}{T'} = \frac{2 \pi R^2 + 2 \pi R H}{2 \pi R'^2 + 2 \pi R' H'} = \frac{R^2 + R H}{R'^2 + R' H'} = \frac{R (R + H)}{R' (R' + H')} \\ = \frac{R}{R'} \times \frac{R + H}{R' + H'}.$$

By theory of proportion if

$R : R' :: H : H'$, then $R + H : R' + H' :: R : R' :: H : H'$.

$$\therefore \frac{T}{T} = \frac{R}{R'} \times \frac{R}{R'} = \frac{R^2}{R'^2} = \frac{H}{H'} \times \frac{H}{H'} = \frac{H^2}{H'^2}.$$

Again:

$$\frac{V}{V'} = \frac{\pi R^2 H}{\pi R'^2 H'} = \frac{R^2}{R'^2} \times \frac{H}{H'} = \frac{R^3}{R'^3} = \frac{H^3}{H'^3}.$$

CONES.

Def. I. A *conical surface* is one generated by a line which moves, always touching a fixed curved line and passes through a fixed point, not in the plane of the curve.

The terms, *generatrix* and *directrix* have the same significance as in the cylindrical surface, and the fixed point is called the *vertex* or *apex*.

Def. II. If the generatrix does not terminate in the fixed point, it generates two portions, called the *upper* and *lower nappes*, respectively.

Def. III. The generatrix in any position is called an *element* of the conical surface.

Def. IV. A *cone* is a solid bounded by a conical surface with a closed curve for directrix, and a section cutting all the elements.

The *base*, *total* and *lateral surface* have the usual meaning.

Def. V. The *altitude* of a cone is the perpendicular distance from the apex to the plane of the base.

Def. VI. A *circular cone* is one having a circular base. The straight line joining the apex and the centre of the base is called the *axis*.

Def. VII. If the axis is perpendicular to the base, it is a *right cone*; otherwise, it is called an *oblique cone*.

Def. VIII. A *cone of revolution* is a right circular cone generated by a right triangle revolving about one of its

legs. The hypotenuse of this triangle in any position is called the *slant height*.

Similar cones of revolution are those generated by similar right triangles.

Def. IX. A *tangent line* to a cone is a straight line, which touches the surface in a single point.

Def. X. A *tangent plane* is one containing one element of the surface, known as the *element of contact*.

Def. XI. A pyramid is said to be *inscribed* in a cone, when its base is inscribed in the base of the cone. A pyramid is *circumscribed* about a cone when its base is circumscribed about the base of the cone and its faces are tangent to the conical surface.

Def. XII. A *truncated cone* is the portion of a cone comprised between the base and any plane section. If the section is parallel to the base it is called a *frustum*.

The *altitude* of a frustum is the perpendicular distance between its bases, and the *slant height* of a frustum of a cone of revolution is the altitude of one of its trapezoid faces.

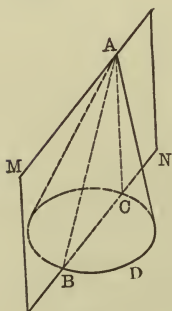


Fig. 53.

Proposition XLVII.

Every section of a cone made by a plane through the vertex is a triangle.

Statement: Let ABC be a section of the cone $A - BCD$ made by the plane MN through A , then ABC is a triangle.

Analysis: The lines of intersection of the plane MN with the conical surface determines the character of the figure. The intersection with the base, which is a plane, we know to be a straight line. Since the elements are straight lines, meeting at A ,

it is only necessary to show that the other two sides are elements.

Proof: Let BC be the intersection of the cutting plane with the base, then it is a straight line. Draw the straight lines AB and AC:

They are elements. (Why?)

They also lie in the cutting plane. (Why?)

$\therefore$ they are intersections of the plane with the conical surface.

$\therefore$ the section ABC is a triangle.

Proposition XLVIII.

Every section of a cone parallel to the base is a circle.

Statement: Let FKH be a section of cone A — EDG $\parallel$ to base EDG. Then FKH is a circle.

Analysis: If FKH is a circle, the points on its circumference are all equally distant from some point within it. If it is a circle it is similar to the base, and the axis of the cone ought to pass through this central point.

By connecting, then, the point in FKH, where the axis cuts it with any two or three points on its circumference and comparing, by the use of lines in the base, the truth of the proposition may be seen.

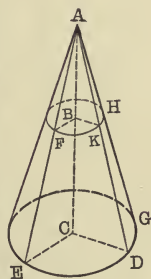


Fig. 54.

Proof: Draw the axis, AC, cutting FKH in B. Draw also the elements AE, AD and AG, touching FKH at F, K and H, respectively. Join B with F and K. Then the triangles ABF and ACE are similar, also ABK and ACD.

(Why?)

$$\therefore FB : EC :: AB : AC \text{ and } BK : CD :: AB : AC,$$

$$\therefore FB : EC :: BK : CD \text{ but } EC = CD$$

(radii of circle EDG.)

$$\therefore FB = BK.$$

Hence FKH is a circle, with centre at B.

The following propositions may be immediately inferred from the fact that a cone is merely a pyramid of an infinite number of faces:—

First. The lateral area of a cone of revolution (corresponding to a regular pyramid) is equal to one-half the product of the slant height by the circumference of the base.

Hence if S represents lateral area, T total area, L the slant height, and R the radius of the base, in a cone of revolution, then

$$S = \frac{1}{2} (2 \pi R \times L) = \pi R \times L.$$

$$T = \pi R \times L + \pi R^2 = \pi R (L + R).$$

Second. The volume of a circular cone equals one-third the product of the altitude by the area of the base; hence, calling V the volume, H the altitude, and R the radius of the base,

$$V = \frac{1}{3} \pi R^2 H.$$

Third. The lateral area of the frustum of a cone of revolution equals one-half the sum of the circumferences of upper and lower base, multiplied by the slant height; or what amounts to the same, the lateral area of such a frustum equals the slant height by the circumference of the mid-section.

Fourth. The volume of the frustum of a circular cone is equivalent to one-third of the altitude multiplied by the sum of the upper base, the lower base, and a mean proportional between them.

Proposition XLIX.

The lateral areas, or the total areas of two similar cones of revolution are to each other as the squares of their altitudes, as the squares of their radii, or as the squares of their slant heights; and their volumes are as the cubes of these like dimensions.

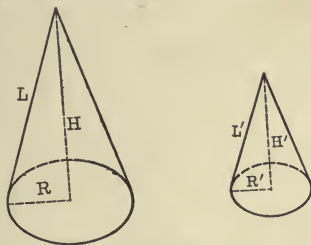


Fig. 55.

Call S and S' , the lateral surfaces; T and T' , the total surfaces; V and V' , the volumes; H and H' , the altitudes; R and R' , the radii; L and L' , the slant heights; then

$$S = \pi RL, \quad S' = \pi R'L', \quad T = \pi R(R + L).$$

$$T' = \pi R'(R' + L'), \quad V = \frac{1}{3} \pi R^2 H$$

$$\text{and} \quad V' = \frac{1}{3} \pi R'^2 H'$$

$$\text{Hence} \quad \frac{S}{S'} = \frac{\cancel{\pi}RL}{\cancel{\pi}R'L'} = \frac{RL}{R'L'} = \frac{R}{R'} \times \frac{L}{L'},$$

$$\frac{T}{T'} = \frac{\pi R(R + L)}{\pi R'(R' + L')} = \frac{R}{R'} \times \frac{R + L}{R' + L'},$$

$$\frac{V}{V'} = \frac{\frac{1}{3} \pi R^2 H}{\frac{1}{3} \pi R'^2 H'} = \frac{R^2}{R'^2} \times \frac{H}{H'}.$$

$$\text{But} \quad \frac{R}{R'} = \frac{H}{H'} = \frac{L}{L'} = \frac{R + L}{R' + L'}.$$

(Since the generating triangles are similar.)

$$\therefore \frac{S}{S'} = \frac{R}{R'} \times \frac{R}{R'} = \frac{R^2}{R'^2} = \frac{H^2}{H'^2} = \frac{L^2}{L'^2}$$

$$\text{and} \quad \frac{T}{T'} = \frac{R^2}{R'^2}, \text{ etc., and } \frac{V}{V'} = \frac{R^3}{R'^3}, \text{ etc.}$$

EXERCISE IV.

1. The total area of a cylinder of revolution is 385 sq. in. The height is four times the radius of the base. Find radius and height.

2. How much sheeting will it take for a smoke stack 60 ft. high and 12 in. in diameter?

3. The altitude of a cone of revolution is 15 in., the radius of its base is 8 in. What is its lateral surface?

4. The sloping face of a conical excavation measures 13 ft., and it is 10 ft. across at the mouth. What is the amount of excavation?

5. A torpedo is made up of a right cylinder, capped by two right cones. The cylindrical part is 4 ft. long and the conical parts each 12 in. high. The diameter is $10\frac{1}{2}$ in. Find contents.

6. The bell-shaped mouth of a cylindrical stack, 14 in. in diameter, is 18 in. long, measured on its sloping face, and its largest diameter is 20 in. How much sheeting will it take?

7. Find the volume of a cone of revolution whose slant height is 26 ft. and whose lateral area is 943 sq. ft.

8. A cubical mass of lead whose total surface is 486 sq. in. is melted and cast into a cone, the diameter of whose base is 7 in. What will be the height of the cone?

9. A tapering discharge pipe $\frac{3}{4}$ in. thick, whose largest diameter is 4 in. and smallest 2 in., is 10 ft. long. How much metal was required for it?

10. How far, respectively, from the vertex of a cone are two sections parallel to the base, if they divide the cone into three equal parts, the altitude of the cone being 9 ft.

11. A right cone is cut into two unequal parts by a plane parallel to the base. The lower part is 7 times the

upper part, and the height of the cone is 12 in. Find distance from the vertex to the plane.

12. A bucket has an upper diameter of 15 in., a bottom diameter of 12 in., and a height of 14 in. Counting 231 cu. in. to a gallon, find the contents in gallons.

13. The lateral area of a cone is 1,232 sq. ft. The slant height is 25 ft. What is the area of a section 6 ft. from the vertex?

14. The radius of the base of a cone is $\frac{3}{4}$ the height, and the lateral surface is 330 sq. in. Find the altitude and the radius.

15. The contents of a cylindrical cistern in gallons (231 cu. in. to the gallon) is given by the formula, $D^2 \times H \times .0034$, where D is the diameter of the base, and H is the altitude. Verify the formula.

16. The total areas of two similar cylinders of revolution are 50 and 72 sq. ft., respectively. The volume of the larger is 1,728 cu. ft. What is the volume of the other?

THE SPHERE.

Definitions.

Def. I. A *sphere* is a solid bounded by a surface, every point of which is equally distant from a point within, called its centre.

Evidently a sphere may be generated by the revolution of a semi-circle about its diameter.

Def. II. The *radius* of a sphere is the distance from its centre to any point on its surface. A *diameter* is a line through the centre limited by the surface. It is equal to twice the radius.

Def. III. A line or plane is *tangent* to a sphere, when it has but one point in common with the sphere.

Def. IV. A *great circle* of a sphere is a circle whose plane divides the sphere into two equal parts. All other circles are known as *small circles*. The plane of a great circle passes through the centre.

Def. V. The *axis* of a circle of a sphere is that diameter of the sphere that is perpendicular to the plane of the circle. Its ends are called *poles* of the circle.

Def. VI. The distance between two points on a sphere is understood to be the arc of a great circle included between them.

This distance is the shortest distance between the two points; hence the designation.

Def. VII. The distance on the spherical surface from the *nearer* pole of a small circle to any point on its circumference is called the circle's *polar distance*.

Solids are said to be inscribed in a sphere, when all their vertices are in the spherical surface.

Proposition L.

All sections of a sphere are circles.

Analysis: The characteristic of a circle is that its points shall all be equally distant from a point within. Since we know that all radii of the sphere are equal, it is suggested to connect these points with the centre of the sphere, and to locate the centre of the section by drawing a diameter perpendicular to it, since it can be inferred from what we know of circles that a diameter perpendicular to this section, will bisect all the lines through its foot, in the section. Complete the proof.

As corollaries, the following may be inferred: —

(a) The line joining the centre of a sphere to the centre of any of its circles, is perpendicular to the plane of the circle.

(b) Circles of a sphere equally distant from its centre are equal.

(c) Of two circles made by planes unequally distant from the centre, the nearer is the greater.

(d) All great circles of a sphere are equal.

(e) Two great circles bisect each other.

(f) If the planes of two great circles are perpendicular each passes through the poles of the other.

(g) Through any three points on the surface of a sphere one circle may be passed. One-fourth of a great circle is called a *quadrant*.

Proposition LI.

A point on the surface of a sphere which is at a quadrant's distance from each of two other points, not the extremities of a diameter, is a pole of the great circle passing through these points.

Statement: Let A on the sphere D, be at a quadrant's distance from B and C, then A is the pole of the great circle BCE through B and C.

Analysis: Since, by definition, the pole of a circle is the end of the diameter perpendicular to its plane, it is immediately suggested to draw through A a diameter and prove it perpendicular to the plane of EBC, remembering that a line is perpendicular to a plane if it is perpendicular to two lines in that plane, drawn through its foot.

Proof: Draw the diameter AF, cutting the plane EBC at D (since the plane of a great circle must pass through the centre of the sphere).

Draw the radii DB and DC. Then it remains to show that $\angle ADB$ and ADC are right angles.

Complete the proof, remembering that arcs AC and AB are quadrants.

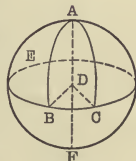


Fig. 56.

Proposition LII.

Given a material sphere, to find its diameter.

Analysis: Since it is impossible to enter the sphere, it is necessary to take points on its surface, and from super-

ficial measurements to construct sections, and by using our knowledge of the relations of lines within a sphere, find our diameter.

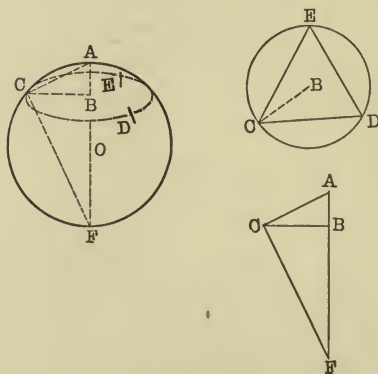


Fig. 57.

it, and with the dividers measure the distances (chords) DC, DE and EC.

Draw the triangle DCE formed of these chords.

Find the centre B of this triangle; B will be the centre of the small circle passing through E, C and D, and BC is its radius.

Construct a right triangle with AC (measured with the dividers) as hypotenuse and BC as one leg; let ABC be this triangle. Draw a perpendicular to AC at C, and produce AB to meet it at F, then AF is the required diameter.

For A is the pole of the circle, ECD, and BC its radius, and since ABF was drawn perpendicular to BC at B, the centre of the circle ECD, it passes through the centre of the sphere.

(Why?)

Also $\angle ACF$ is a right angle and hence inscribed in a semi-circle ; hence its sides AC and CF terminate at the ends of the diameter at A and F .

The same thing may be shown by comparison with a section of the sphere through its diameter.

The triangle ACF is then seen to be identical with its prototype ACF in the sphere.

Proposition LIII.

A plane perpendicular to a radius at its extremity is tangent to the sphere at that point.

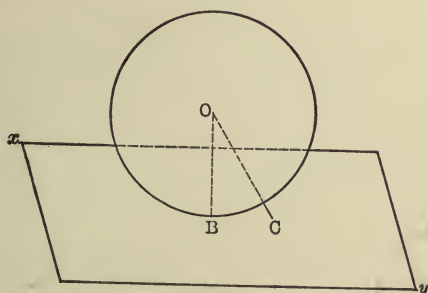


Fig. 58.

Statement: If the plane xy is perpendicular to a radius OB of the sphere O , at B , it is tangent to the sphere at B .

Analysis: Since a plane is tangent when it has but one point in common with the sphere, if a line, other than the radius OB , be drawn from O to any point of xy other than B , it can be shown that it is longer than OB and hence outside the sphere. Complete the proof.

Cor. I. A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.

Cor. II. A line tangent to a sphere at a given point,

lies in the plane tangent to the sphere at that point, and conversely.

Cor. III. The plane of two lines tangent to the sphere at the same point is tangent to the sphere at that point.

Proposition LIV. (Problem.)

To inscribe a sphere in any given tetrahedron.

Analysis: That a sphere may be inscribed in the tetrahedron $A - BCD$, its centre must be so determined that it shall be equally distant from the four faces. In that connection, the locus of points equally distant from the

faces of a dihedral angle is suggested.

Construction: Bisect the dihedral angles BD , CA and AB by planes, intersecting at O . Then O will be the required centre.

Proof: The plane BOD is the locus of all points equally distant from the

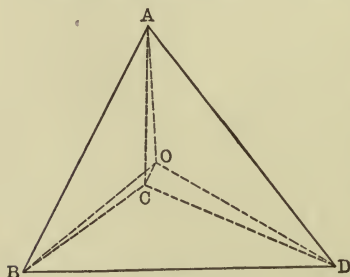


Fig. 59.

faces ABD and BCD , the plane OCA is the locus of all points equally distant from the faces ACD and ACB , and the plane OAB is the locus of all points equally distant from the faces ABD and ACB . Therefore, the points O , where these planes intersect, is equally distant from all the faces.

The planes must intersect in a common point as O , because O is the only point that possesses the characteristic property of all the bisecting planes.

Proposition LV. (Problem.)

To circumscribe a sphere about any tetrahedron.

Analysis: Since the centre of such a sphere must be equally distant from the four vertices of the tetrahedron, it is suggested that a perpendicular to a plane is the locus of all points equally distant from points in the plane, equally distant from its foot. Hence a perpendicular to the plane of any of the triangular faces at its centre fulfills the condition.

Complete the proof.

Proposition LVI.

The intersection of two spherical surfaces is the circumference of a circle whose plane is perpendicular to the line joining the centres of the surfaces and whose centre is in that line.

Statement: If the surfaces of the spheres A and B intersect, to prove that the line of intersection is the circumference of a circle, whose centre lies in the line AB.

Analysis: Since a sphere may be generated by revolving a semi-circle about its diameter, the

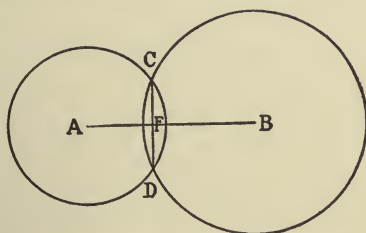


Fig. 59a.

relations between the lines can be determined by making a section through the two centres.

Proof: Pass a section through the two centres, intersecting the two surfaces in the great circles A and B, which intersect at C and D. Join C and D by a straight line also A and B.

By Plane Geometry AB is perpendicular to CD at the middle point F. Hence if this whole plane figure be revolved about AB as an axis, the points C (or D) will describe a circle with centre at F on the line AB.

Definitions.

The angle formed by two intersecting curves is the *angle formed by the tangents* to these curves at the point of intersection.

If the curves are arcs of great circles, the angle is called a *spherical angle*.

A spherical polygon is a portion of the surface of a sphere, bounded by arcs of great circles. These spherical polygons are named triangles, quadrilaterals, pentagons, etc., according to the number of sides, just as are plane polygons.

The nomenclature is exactly the same for the parts of a spherical polygon, as for plane polygons. It is to be borne in mind, only, that the lines are all great circle arcs, and hence are usually expressed in circular units, not in linear units. It is to be understood, that a great circle arc bears exactly the same relation to a spherical surface that a straight line does to a plane surface. In fact a great circle arc is really a straight line bent to conform to a spherical surface. In consequence a great circle arc can be readily drawn on a sphere with a straight ruler, if it is flexible enough to conform to the surface. Observe that a great circle arc is the shortest distance between two points on a sphere, and compare this fact with the similar character of a straight line.

Proposition LVII.

A spherical angle is measured by the arc of the great circle described with its vertex as pole and included between its sides (or sides produced).

Statement: The spherical angle BAC is measured by arc BC, described from pole A.

Analysis: In determining the value of the angle A, by definition, the angle between the tangents EA and FA, drawn to the arcs BA and CA, respectively, decides. By comparing this with a central angle, which BC subtends, the arc BC is introduced.



Fig. 60.

Proof: The spherical angle $A = \angle EAF$ between the tangents EA and FA (by definition); also the diameter AOD, drawn from A, is perpendicular to the plane of the circle BCH, of which BC is an arc (Why?). Hence it is perpendicular to OB and OC, radii of BCH. But EA and FA are both perpendicular to AO (Why?). Hence EA and AF are $\parallel$ respectively to BO and OC, since they lie in the same planes. $\therefore \angle EAF = \angle BOC$, and $\angle BOC$ is measured by arc BC. Hence $\angle EAF = \text{arc BC}$.

Cor. A spherical angle has the same measure as the dihedral angle formed by the planes of its sides.

Proposition LVIII.

One side of a spherical triangle is less than the sum of the other two.

Statement: Let bac be a spherical triangle on the surface of the sphere O, and ac the largest side say; then $ac < ab + bc$.

Analysis: The sides of a spherical triangle being arcs of circles are measured by the angle at the centre of the sphere that they subtend.

It is evident that if the three vertices of a spherical triangle be connected to the centre of the sphere by radii,

these radii will form the edges of a trihedral angle at O , whose faces will be the planes of the sides.

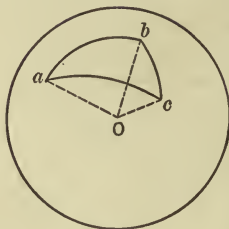


Fig. 61.

Recalling the proposition referring to the relative size of one face angle of a trihedral to the sum of the other two, complete the proof.

Exercise: Prove also by the use of the polyhedral angle at the centre of the sphere, that the sum of the

sides of a spherical polygon is less than four right angles.

Definition: If with the vertices of a spherical triangle as poles intersecting arcs of great circles be described, they will form another spherical triangle. These two triangles are called *polar triangles* with reference to each other.

If with the vertices of a spherical triangle, as poles complete great circles are described, they will, by their intersection, divide the surface of the sphere into eight spherical triangles.

If the planes of these circles are mutually perpendicular the surface is divided into eight trirectangular * triangles. Each of these triangles is divided into 90 spherical degrees as they are called; that is, the entire surface of the sphere is divided into 720 spherical degrees. These degrees are used as units of spherical surface. A spherical degree must be carefully distinguished from ordinary degrees, as spherical degrees are small unit surfaces.

Definition: The *spherical excess* is the name given to the number of degrees by which the sum of the angles of a spherical triangle exceeds 180° . Evidently the trirectangular triangles might be divided into any number of parts,

* See remark under Proposition LX.

as 90 is purely arbitrary, but it has been found most convenient to make the number of parts 90.

Proposition LIX.

In two polar triangles each angle of one is the supplement of the opposite side in the other.

Statement: If ABC is a spherical triangle and $A'B'C'$ its polar, then denoting the sides opposite the angles, by the small letters, corresponding to the large letters representing the angle.

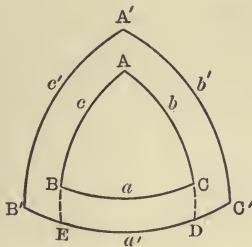


Fig. 62.

$$A + a' = 180 \quad B + b' = 180, \text{ etc.}$$

$$\text{Also } A' + a = 180 \quad B' + b = 180, \text{ etc.}$$

Analysis: As usual when two figures are to be compared they are joined together in some simple, direct way. The plan that is readily suggested here, is to produce the sides of the inner triangle ABC to meet the side, say $B'C'$, of $A'B'C'$.

Proof: Let AB and AC be produced (along a great circle of course) to meet $B'C'$ at E and D , respectively. Then $B'D$ and $C'E$ are quadrants, hence,

$$B'D + C'E = 180^\circ, \text{ but } B'D = B'E + ED$$

$$\text{and } C'E = C'D + ED.$$

$$\therefore B'E + ED + C'D + ED = 180$$

$$(B'E + ED + C'D = B'C').$$

$$\therefore B'C' + ED = 180.$$

But ED measures the angle A (by Prop. LVII) and $B'C' = a'$, hence $a' + A = 180$. In the same way by extending the other sides the other relations expressed above may be shown.

Proposition LX.

The sum of the angles of a spherical triangle is greater than 180° and less than 540° .

Statement: To prove that $\angle A + \angle B + \angle C$ is $> 180^\circ$ and $< 540^\circ$ in triangle ABC.

Analysis: Recalling that the sum of the sides of any spherical triangle is less than 360° ; and also remembering the relation between the sides and angles of polar triangles, the employment of the triangle polar to ABC is immediately suggested.

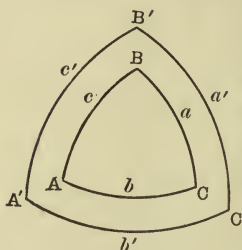


Fig. 63.

Proof: Construct the triangle $A'B'C'$, polar to ABC. Represent as usual the sides by the small letters corresponding to the large angle-letters opposite. Then (by Prop. LIX)

$$A + a' = 180^\circ$$

$$B + b' = 180^\circ$$

$$C + c' = 180^\circ$$

Add; $(A + B + C) + (a' + b' + c') = 540^\circ$;
transposing $(A + B + C) = 540^\circ - (a' + b' + c')$.

But $a' + b' + c' < 360^\circ$ (by Exercise under Prop. LVIII).

$$\therefore A + B + C = 540^\circ \text{ minus a quantity less than } 360^\circ.$$

$$\therefore A + B + C > 180^\circ \text{ (since } 540^\circ = 360^\circ + 180^\circ \text{)}.$$

$a' + b' + c'$ cannot be 0, or there would be no triangle.

$$\therefore A + B + C < 540^\circ.$$

Remark: Unlike plane triangles spherical triangles may have more than one right angle. Draw one with three right angles. If a spherical triangle has two right angles, it is called *bi-rectangular*; if it has three it is called *tri-rectangular*.

Definitions.

The propositions already proved for plane triangles can be verified in the case of spherical triangles by identical processes, in particular by superposition. For instance, two triangles on the same sphere or equal spheres are equal if two sides and the included angle in one are equal to two sides and the included angle in the other, etc.

A *zone* is a portion of the *surface* of a sphere included between two parallel planes. The circumferences of the sections are called *bases* and the perpendicular between the planes is the *altitude*.

If a sphere is cut by a single plane, either portion of the surface, so divided, is called a *zone of one base*.

A *lune* is a portion of the surface of a sphere, bounded by the semi-circumferences of two great circles. The angle between these two semi-circumferences is called the *angle of the lune*.

Proposition LXI.

The area of the surface generated by a straight line revolving about an axis in its plane is equal to the product of the projection of the line on the axis by the circumference, whose radius is a perpendicular erected at the middle point of the line and terminated by the axis.

Statement: Let BF be the line revolving about the axis *xy*, KH its projection on *xy*, and CE the perpendicular at its mid-point, terminating in *xy*, then

$$\text{surface BF} = \text{KH} \times 2\pi\text{CE}.$$

Analysis: There are evidently three cases: when the line BF is $\parallel$ to *xy*; when it is not parallel, but does not intersect *xy*; and when it does meet *xy*.

In the first case the surface generated is clearly that of

a right cylinder, and the perpendicular to the middle of the line is the same as the radius of the bases. Fig. 64 (M).

In the other cases it is necessary to determine the kind of surface generated; and having decided that, apply the

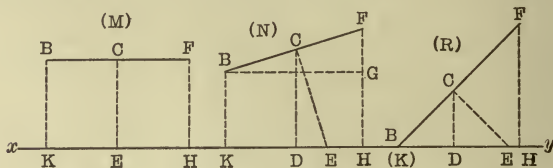


Fig. 64.

corresponding rule for area; then by similar triangles reduce the expression obtained for area to the expression required by the proposition.

Proof: Case II. Fig. 64 (N).

The surface is that of a frustrum of a cone. Hence surface $BF = BF \times 2\pi CD$, where CD is the radius of the mid-section.

Draw $BG \parallel$ to xy and $\perp$ to FGH , the radius of larger base of frustrum.

Then BFG and DCE are similar right triangles. (Why?)

$$\therefore BF : CE :: BG : CD.$$

But $BG = KH$ (Why?). $\therefore BF : CE :: KH : CD$.

Multiplying extremes and means:

$$BF \times CD = CE \times KH. \quad (1).$$

Surface $BF = BF \times 2\pi CD$ may be expressed thus:

$$\text{Surface } BF = BF \times CD \times 2\pi. \quad (2),$$

Substituting value of $BF \times CD$ from (1) in (2),

$$\text{Surface } BF = CE \times KH \times 2\pi = KH \times 2\pi CE.$$

Proof: Case III.

Here B coincides with K and the surface is that of a cone.

Hence surface $BF = \frac{1}{2} (BF \times 2\pi FH) = BF \times \pi FH(a)$

The triangles CDE and BFH are similar. (Why?)

$\therefore$ $BF : CE :: BH : CD$;
 hence $BF \times CD = CE \times BH$,
 but $CD = \frac{1}{2} FH$ or $FH = 2CD$;
 hence $BF \times \frac{1}{2} FH = CE \times BH$
 or $BF \times FH = 2CE \times BH$ (b).

Substituting this value of $BF \times FH$ from (b) in (a),

Surface $BF = BH \times 2\pi CE$ or $KH \times 2\pi CE$, since B and K are same point.

Proposition LXII.

The area of the surface of a sphere is equal to the product of its diameter by the circumference of a great circle.

Statement: If O be a sphere of diameter mt , then surface of O = $mt \times 2\pi mo$.

Analysis: Since any curve may be regarded as made up of an infinite number of infinitely small straight lines, and hence a circle may be considered as a polygon of infinite number of sides, taking a section through the centre of the sphere O, we may inscribe in the section (which is a circle) a regular polygon of any number of sides. If then half of this polygon is revolved about the diameter as an axis, each side will generate a surface, which can be determined by Prop. LXI.

Proof: Inscribe in the semi-circle $mabt$ the half of a

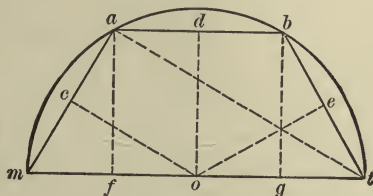


Fig. 65.

regular hexagon (a hexagon is the easiest regular polygon

to inscribe) and from the vertices a and b drop perpendiculars to determine the projections of the sides ma , ab and bt on the diameter mt ; also

Draw co perpendicular to ma at its middle point c ,

Draw do perpendicular to ab at its middle point d ,

Draw oe perpendicular to bt at its middle point e .

Then mf is the projection of ma on mt .

Then fg is the projection of ab on mt ,

and gt is the projection of bt on mt .

If the figure is revolved about mt as an axis, the semi-circle will describe the sphere O , and each side of the semi-polygon will describe an area, under the conditions of Prop. LXI.

$$\therefore \text{area described by } ma = mf \times 2\pi co$$

$$\text{area described by } ab = fg \times 2\pi do$$

$$\text{area described by } bt = gt \times 2\pi oe$$

$$\text{add; } \frac{\text{area described by } (ma + ab + bt) =$$

$(mf + fg + gt) \times 2\pi co$ [since $co = do = oe$], that is, area described by $mabt = mt \times 2\pi co$ [since $mf + fg + gt = mt$].

If now the sides of the polygon be indefinitely increased in number, the sum of their projections will continue to be mt , since the extremities m and t may remain as they are and co will become longer and longer until it equals the radius in length, when the semi-perimeter of the polygon becomes identical with the semi-circumference.

But the above equation is not confined to any particular polygon, for, although the original polygon is a hexagon, the fact that it has six sides has not been considered in the conditions that formed the equation. Hence it is true for any inscribed polygon, even one which has an infinite number of sides.

$\therefore$ area described by semi-polygon of infinite number of sides $= mt \times 2\pi R$,
 or area of sphere $= mt \times 2\pi R = 2R \times 2\pi R = 4\pi R^2$,
 since a polygon of an infinite number of sides is a circle, and a semi-circle describes a spherical surface, if revolved about its diameter.

Cor. I. Surfaces of spheres are to each other as the squares of their radii.

Cor. II. The area of a zone equals the product of its altitude by the circumference of a great circle. For (Fig. 65) the area described by arc $ab = \text{zone } ab = gf \times 2\pi R$, etc.

Again, zone described by arc

$$ma = mf \times 2\pi R = mf \times mt \times \pi.$$

But $mf \times mt = \overline{ma}^2$ in right triangle mat
 (by Plane Geometry).

$\therefore$ zone described by $ma = \pi \overline{ma}^2$.

That is, the area of a zone of one base equals the area of a circle whose radius is the chord of the generating arc.

Cor. III. Zones on the same sphere or equal spheres are to each other as their altitudes.

Proposition LXIII.

The area of a lune, in spherical degrees, equals twice its angle.

Statement: The area of the lune ABCD on the sphere O equals $2A$, in spherical degrees.

Analysis: It is evidently necessary to compare the surface of the lune with the entire spherical surface, and the form of the lune suggests that it can be best compared by its angle, or what amounts to the same, by the arc which measures its angle.

Proof: With A as a pole describe the great circle BCE,

intersecting the side of the lune in B and C. Then the angle A is measured by the arc BC. (Why?)

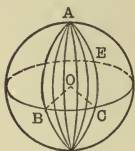


Fig. 66.

If arc BC is commensurable with the circle BCE, say BC contains a measure, m times, and the whole circle BCE, n times, then $BC : BCE :: m : n$ (1). Through these division points and A, draw great circles; they will divide the whole surface into n unit-lunes and the lune ABCD into m unit-lunes.

$\therefore$ If L represent the area of the lune and S the spherical surface, $L : S :: m : n$ (2) $\therefore$ from (1) and (2) $L : S :: BC : BCE$.

But $BC = \angle A$ (Prop. LVII),

$$BCE = 360^\circ$$

and

$$S = 720 \text{ spherical degrees.}$$

$$\therefore L : 720 :: A : 360,^*$$

that is, $L = \frac{720 A}{360} = 2A$ spherical degrees.

If BC and BCE are incommensurable, the proposition can be proved as usual, by reducing the unit.

Proposition LXIV.

The area of a spherical triangle is equal to its spherical excess, expressed in spherical degrees.

Statement: If ABC is a spherical triangle, then area $ABC = A + B + C - 180 = E$, in spherical degrees.

Analysis: Since we know the area of lunes, it would be helpful, evidently, to express the triangle in lunes. To accom-



Fig. 67.

* Since A and 360 are both expressed in ordinary degrees, their ratio, $A : 360$, is an absolute number, like 4, 6, etc.

plish this it is necessary to complete the great circles, of which the sides of the triangles are arcs.

Proof: Complete the great circles as indicated. They will intersect in the vertices A, B, and C, and also in the points A', B', and C'.

$$\text{Then } BCA + BCA' = \text{lune } ABA'C$$

$$BCA + BAC' = \text{lune } CBC'A$$

$$BCA + CAB' = \text{lune } CBAB'$$

$$\text{add; } \quad \overline{2BCA + BCA + BCA' + BAC' + CAB'} \\ = \text{lunes } ABA'C + CBC'A + CBAB'.$$

$$\text{But } BCA + BCA' + BAC' + CAB' = \text{the hemisphere} \\ (\text{since } BCA' = B'C'A)^*$$

$$\text{and lunes } ABA'C + CBC'A + CBAB' = 2A + 2C + 2B$$

$$\text{Hence, since the hemispheres} = 360 \text{ spherical degrees,}$$

$$BCA + 180 = A + B + C \text{ spherical degrees,}$$

$$\text{or } BCA = A + B + C - 180 = E \text{ spherical degrees.}$$

Proposition LXV.

The volume of a sphere equals one-third the radius multiplied by the surface.

Statement: Let X be a sphere of radius R and surface S, then the volume $V = \frac{1}{3}RS$.

Analysis: The proposition suggests the volume of a pyramid, and regarding the surface of the sphere as made up of an infinite number of plane faces, and connecting the intersection points of these planes with the centre of the sphere, the analogy is complete.

* B'C'A is called the symmetrical triangle to BCA', formed by drawing diameters through B, C, and A', and joining the other extremities of these diameters by arcs of great circles. Clearly the sides of these triangles are equal in length, but arranged in reverse order, hence they are equivalent.

Proof: Circumscribe a polygon of any convenient number of faces about the sphere.

Join the vertices of this polyhedron with the centre of the sphere, and then pass planes through these centre lines and the edges of the polyhedron. The polyhedron is thus divided into pyramids with their apices at the centre; and with the faces of the polyhedron as bases. Since these bases are tangent to the sphere the radius of the sphere will be the altitude of each pyramid. (Why?)

The volume of each pyramid equals $\frac{1}{3}$ its base, by the radius, hence the sum of all the pyramids will equal $\frac{1}{3}$ the radius times the surface of the polyhedron (which is the sum of all the bases). If the number of these faces is indefinitely increased, the surface of the polyhedron will become eventually the surface of the sphere, but the volume is always equal to $\frac{1}{3}$ the radius times the surface, hence, when it coincides with the sphere, $V = \frac{1}{3} RS$.

Cor. I. The volume of the sphere may be stated thus: Let R = radius; S , the surface; and V , the volume.

By Prop. LXV, $V = \frac{1}{3} RS$,
 but $S = 4\pi R^2$ (by Prop. LXII),
 $\therefore V = \frac{4}{3} \pi R^3 = \frac{1}{6} \pi D^3$,
 since $R = \frac{1}{2} D$, hence $R^3 = \frac{1}{8} D^3$.

Cor. II. The volumes of two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.

Cor. III. The volume of a spherical sector is equal to one-third the product of the radius and the area of the zone forming the base of the sector.

EXERCISE V.

1. The radius of a sphere is 12 in. Find the area of the section made by a plane bisecting the radius perpendicularly.

2. What is the locus of the centres of spheres having a given radius and tangent to a given sphere?

3. Find the locus of the centres of spheres having a given radius and tangent to two given spheres.

4. If two straight lines are tangent to a sphere at the same point, their plane is tangent to the sphere at that point.

5. To draw a sphere tangent to four intersecting planes.

6. Show how to pass a sphere through four points in space not in the same plane.

7. What is the radius of a sphere whose surface is 25,434 square metres?

8. Two leaden spheres of 20 in. and 40 in. diameter, respectively, are to be melted and a single sphere formed. What is the diameter of this sphere?

9. If specific gravity equals weight divided by volume, what is the diameter of an iron sphere whose weight is 100 lbs. The specific gravity of iron being 7.2.

10. How far can an observer on the ocean see if he is elevated 150 ft. above the surface, taking the earth's radius to be 3,960 miles?

11. How many cubic feet of masonry will it take to build a dome in the form of a half-sphere whose diameter is 21 ft. and thickness 15 in.?

12. Through a given line to pass a plane tangent to a given sphere.

13. To inscribe a sphere in a given tetrahedron.

14. What is the radius of a sphere inscribed in a regular tetrahedron, each of whose edges is $3\sqrt{6}$?

15. On a sphere whose radius is 10 in., a small circle of radius $5\sqrt{3}$ is described. How far is its plane from the centre?

16. Show how to circumscribe a circle about a spherical triangle.

17. The diameter of a sphere is 17 in. From any point on the surface as a pole, with the points of the compass spread 8 in., a circle is described on the spherical surface. Find its area.

18. Find the radius of the section of a sphere 13 in. in diameter made 5 in. from the centre.

19. Construct a spherical surface with a given radius that passes through two given points and touches a given sphere.

20. The angles of a spherical triangle are 125° , 110° and 146° . What is its area in square feet if the radius of the sphere is 7 ft.

21. What is the area of the circle of intersection of two spheres of 8 in. and 6 in. radius, respectively, if their centres are 5 in. apart? -

22. The radii of two parallel sections of a sphere are 6 in. and 4 in., respectively, and they are 2 in. apart. What is the radius of the sphere?

23. How high must a man be raised above the earth's surface that he may see $\frac{1}{8}$ of it?

24. What is the radius of a sphere whose volume is equal numerically to the circumference of a great circle?

25. What is the surface in square feet of a lune whose angle is 36° on a sphere whose radius is 14 ft.?

26. Find the area of the torrid zone on the earth, if its altitude is 3,200 miles, the earth's diameter being 7,920 miles.

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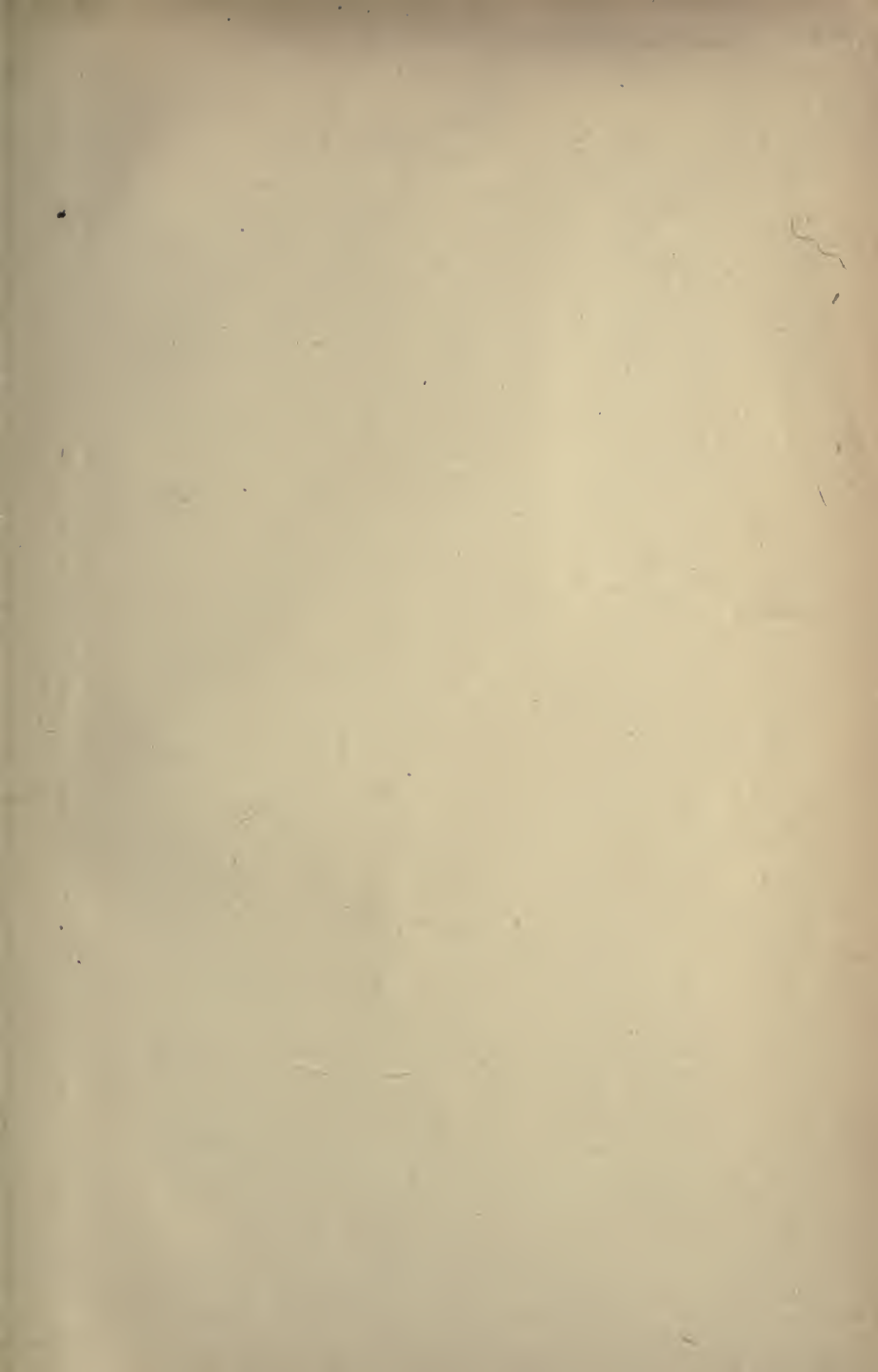
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